Exercises for the Feynman Lectures on Physics by Richard Feynman, Et Al. Chapter 36 Fourier Analysis of Waves– Detailed Work by James Pate Williams, Jr. BA, BS, MSwE, PhD

From *Exercises for the Feynman Lectures on Physics* by Richard Feynman, Robert Leighton, Matthew Sands, et al. **36 *Fourier Analysis of Waves***. Refer to *The Feynman Lectures on Physics Vol. I*, Chapter 50.

Fourier series over a period P where x is contained in the half-open interval [x_0, P).

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty}\left[a_n \cos\left(\frac{2\pi n x}{P}\right) + b_n \sin\left(\frac{2\pi n x}{P}\right)\right]$$

$$a_n = \frac{2}{P}\int_{x_0}^{x_0+P} f(x)\cos\left(\frac{2\pi n x}{P}\right)dx$$

$$b_n = \frac{2}{P}\int_{x_0}^{x_0+P} f(x)\sin\left(\frac{2\pi n x}{P}\right)dx$$

Euler's equation used to derive trigonometric identities.

$$e^{i(\vartheta\pm\varphi)} = \cos(\vartheta\pm\varphi) + i\sin(\vartheta\pm\varphi) = e^{i\vartheta}e^{\pm i\varphi} = (\cos\vartheta + i\sin\vartheta)(\cos\varphi \pm i\sin\varphi) = \cos\vartheta\cos\varphi \mp \sin\vartheta\sin\varphi + i(\cos\vartheta\sin\varphi \pm \sin\vartheta\cos\varphi)$$

$$\cos(\vartheta\pm\varphi) = \cos\vartheta\cos\varphi \mp \sin\vartheta\sin\varphi$$

$$\sin(\vartheta\pm\varphi) = \cos\vartheta\sin\varphi \pm \sin\vartheta\cos\varphi$$

$$2\cos\vartheta\cos\varphi = \cos(\vartheta-\varphi) + \cos(\vartheta+\varphi)$$

$$2\sin\vartheta\cos\varphi = \sin(\vartheta+\varphi) - \sin(\vartheta-\varphi)$$

$$e^{ix}e^{-ix} = e^0 = 1 = (\cos x + i\sin x)(\cos x - i\sin x) = (\cos x)^2 + (\sin x)^2$$

$$e^{2ix} = \cos 2x + i\sin 2x = e^{ix}e^{ix} = (\cos x + i\sin x)(\cos x + i\sin x) = (\cos x)^2 - (\sin x)^2 + i(\cos x\sin x + \sin x\cos x)$$

$$\cos 2x = (\cos x)^2 - (\sin x)^2 = 1 - (\sin x)^2 - (\sin x)^2 = 1 - 2(\sin x)^2$$

$$(\sin x)^2 = \frac{1}{2}(1 - \cos 2x)$$

$$\sin 2x = 2\cos x \sin x$$

36.1 (a) y(x) = const. (b) y(x) = sin x 0 <= x <= 2 * pi.

(a)

$$y(x) = C \; \forall x \in [0, P]$$

$$a_n = \frac{2C}{P}\int_{x_0}^{x_0+P} \cos\left(\frac{2\pi n x}{P}\right) dx$$

$$a_n = \frac{2C}{P} \times \frac{P}{2\pi n} \times \sin\left(\frac{2\pi n x}{P}\right)\Bigg]_{x=x_0}^{x=x_0+P} = \frac{C}{\pi n}\sin\left[\frac{2\pi n(x_0+P)}{P}\right] - \frac{C}{\pi n}\sin\left(\frac{2\pi n x_0}{P}\right)$$

$$\sin\left[\frac{2\pi n(x_0+P)}{P}\right] = \cos\left(\frac{2\pi n x_0}{P}\right)\sin(2\pi n) + \sin\left(\frac{2\pi n x_0}{P}\right)\cos(2\pi n) = \sin\left(\frac{2\pi n x_0}{P}\right)$$

$$a_n = \frac{C}{\pi n}\left[\sin\left(\frac{2\pi n x_0}{P}\right) - \sin\left(\frac{2\pi n x_0}{P}\right)\right] = 0$$

$$b_n = \frac{2C}{P}\int_{x_0}^{x_0+P} \sin\left(\frac{2\pi n x}{P}\right) dx$$

$$\begin{aligned} b_n = -\frac{2C}{P} \times \frac{P}{2\pi n} \times \cos\left(\frac{2\pi n x}{P}\right)\Bigg]_{x=x_0}^{x=x_0+P} &= -\frac{C}{\pi n}\left\{\cos\left[\frac{2\pi n(x_0+P)}{P}\right] - \cos\left(\frac{2\pi n x_0}{P}\right)\right\} \\ &= -\frac{C}{\pi n}\left[\cos\left(\frac{2\pi n x_0}{P}\right)\cos(2\pi n) - \sin\left(\frac{2\pi n x_0}{P}\right)\sin(2\pi n) - \cos\left(\frac{2\pi n x_0}{P}\right)\right] \\ &= -\frac{C}{\pi n}\left[\cos\left(\frac{2\pi n x_0}{P}\right) - \cos\left(\frac{2\pi n x_0}{P}\right)\right] = 0 \end{aligned}$$

$$\frac{a_0}{2} = C \therefore a_0 = 2C$$

(b)

$$y(x) = \sin x \; \forall x \in [0, 2\pi] \therefore x_0 = 0, P = 2\pi$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} \sin x \, dx = -\frac{1}{\pi}\cos x]_{x=0}^{x=2\pi} = -\frac{1}{\pi}[\cos(2\pi) - \cos(0)] = -\frac{1}{\pi}(1-1) = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \sin x \cos nx \, dx = \frac{1}{2\pi}\left[\int_0^{2\pi} \sin(x+nx)dx - \int_0^{2\pi} \sin(x-nx)dx\right]$$

$$= \frac{1}{2\pi}\left\{\int_0^{2\pi} \sin[(1+n)x] \, dx - \int_0^{2\pi} \sin[(1-n)x] \, dx\right\}$$

$$= \frac{1}{2\pi}\left\{-\frac{\cos[(1+2n)x]}{1+2n}\Bigg]_{x=0}^{x=2\pi} - \frac{\cos[(1-2n)x]}{1-2n}\Bigg]_{x=0}^{x=2\pi}\right\} =$$

$$= \frac{1}{2\pi}\left\{-\frac{\cos[(1+2n)x]}{1+2n}\Bigg]_{x=0}^{x=2\pi} + \frac{\cos[(1-2n)x]}{2n-1}\Bigg]_{x=0}^{x=2\pi}\right\} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \sin x \sin nx \, dx = 0 \; \forall n > 1$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} (\sin x)^2 dx = \frac{1}{2\pi}\int_0^{2\pi} (1-\cos 2x)dx = 1$$

36.2

$$f(x) = \begin{cases} +1 \; \forall x \in [0,\pi] \\ -1 \; \forall x \in (\pi, 2\pi) \end{cases}$$

$$a_n = \frac{1}{\pi}\left(\int_0^{\pi} \cos nx \, dx - \int_{\pi}^{2\pi} \cos nx \, dx\right) = \frac{1}{\pi n}\left(\sin nx]_{x=0}^{x=\pi} - \sin nx]_{x=\pi}^{x=2\pi}\right) = 0$$

$$b_n = \frac{1}{\pi}\left(\int_0^{\pi} \sin nx \, dx - \int_{\pi}^{2\pi} \sin nx \, dx\right) = -\frac{1}{\pi n}\left(\cos nx]_{x=0}^{x=\pi} - \cos nx]_{x=\pi}^{x=2\pi}\right)$$

$$= -\frac{1}{\pi n}[\cos n\pi + \cos n\pi - 2]$$

$$b_{2n+1} = -\frac{4(-1)^{2n+1}}{\pi(2n+1)} = \frac{4(-1)^{2n+2}}{\pi(2n+1)} = \frac{4}{\pi(2n+1)}$$

$$f(x) = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\sin[(2n+1)x]}{(2n+1)}$$

(a)

$$f\left(\frac{\pi}{2}\right) = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\sin[(2n+1)\,\pi/2]}{(2n+1)}$$

$$\sin[(2n+1)\,\pi/2] = \sin(n\pi + \pi/2) = \cos n\pi \sin(\pi/2) + \sin n\pi \cos(\pi/2) = (-1)^n$$

$$1=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}=\frac{\pi}{4}$$

(b)

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)}=\left(\frac{\pi}{4}\right)^2=\frac{\pi^2}{16}$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+m}}{(2n+1)(2m+1)}=\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{16}$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}$$

(c)

$$\sum_{n=0}^{\infty}4^{-n}=\frac{1}{1-\frac{1}{4}}=\frac{4}{3}$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\sum_{m=0}^{\infty}4^{-m}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(2n+1)^2 4^m}=\frac{4}{3}\times\frac{\pi^2}{8}=\frac{\pi^2}{6}$$

36.3

$$g(x)=\begin{cases}\dfrac{x}{\pi}\ \forall x\in[0,\pi)\\ 1+\dfrac{\pi-x}{\pi}\ \forall x\in[\pi,2\pi)\end{cases}$$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}g(x)\,dx=\frac{1}{\pi^2}\int_0^{\pi}x\,dx+\frac{1}{\pi}\int_{\pi}^{2\pi}dx+\frac{1}{\pi}\int_{\pi}^{2\pi}dx-\frac{1}{\pi^2}\int_{\pi}^{2\pi}x\,dx$$

$$=\frac{\pi^2}{2\pi^2}+\frac{2}{\pi}(2\pi-\pi)-\frac{1}{2\pi^2}[(2\pi)^2-\pi^2]=\frac{1}{2}+2-\frac{3}{2}=1$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}g(x)\cos nx\,dx=\frac{1}{\pi^2}\int_0^{\pi}x\cos nx\,dx+\frac{1}{\pi}\int_{\pi}^{2\pi}\cos nx\,dx+\frac{1}{\pi}\int_{\pi}^{2\pi}\cos nx\,dx-\frac{1}{\pi^2}\int_{\pi}^{2\pi}x\cos nx\,dx$$

$$d(uv)=udv+vdu$$

$$\int d(uv)=uv=\int udv+\int vdu$$

$$\int u dv = uv - \int v du$$

$$u(x) = x$$

$$dv = \cos nx\, dx$$

$$\int \cos nx\, dx = \frac{1}{n} \sin nx$$

$$\int_0^{\pi} x \cos nx\, dx = \frac{x}{n} \sin nx \Big]_{x=0}^{x=\pi} - \frac{1}{n} \int_0^{\pi} \sin nx\, dx = \frac{1}{n^2} \cos nx]_{x=0}^{x=\pi} = \frac{1}{n^2} (\cos n\pi - 1) = \frac{1}{n^2} [(-1)^n - 1]$$

$$= -\frac{2}{(2n+1)^2}$$

$$\int_{\pi}^{2\pi} x \cos nx\, dx = \frac{x}{n} \sin nx \Big]_{x=\pi}^{x=2\pi} - \frac{1}{n} \int_{\pi}^{2\pi} \sin nx\, dx = \frac{1}{n^2} \cos nx]_{x=\pi}^{x=2\pi} = \frac{1}{n^2} (\cos 2\pi n - \cos n\pi)$$

$$= \frac{1}{n^2} [1 - (-1)^n] = \frac{2}{(2n+1)^2}$$

$$a_{2n+1} = -\frac{2}{(2n+1)^2 \pi^2} - \frac{2}{(2n+1)^2 \pi^2} = -\frac{4}{(2n+1)^2 \pi^2}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} g(x) \sin nx\, dx = \frac{1}{\pi^2} \int_0^{\pi} x \sin nx\, dx + \frac{1}{\pi} \int_{\pi}^{2\pi} \sin nx\, dx + \frac{1}{\pi} \int_{\pi}^{2\pi} \sin nx\, dx - \frac{1}{\pi^2} \int_{\pi}^{2\pi} x \sin nx\, dx$$

$$u(x) = x$$

$$dv = \sin nx\, dx$$

$$\int \sin nx\, dx = -\frac{1}{n} \cos nx$$

$$\int_0^{\pi} x \sin nx\, dx = \frac{x}{n} \cos nx \Big]_{x=0}^{x=\pi} - \frac{1}{n} \int_0^{\pi} \cos nx\, dx = \frac{2\pi(-1)^n}{n}$$

$$\int_{\pi}^{2\pi} x \sin nx\, dx = \frac{x}{n} \cos nx \Big]_{x=\pi}^{x=2\pi} - \frac{1}{n} \int_{\pi}^{2\pi} \cos nx\, dx = \frac{2\pi}{n} - \frac{2\pi(-1)^n}{n}$$

$$b_n = \frac{1}{n\pi^2} [(-1)^n + (-1)^n - 2] - \frac{2}{n\pi} (\cos 2\pi n - \cos n\pi) = \frac{1}{n\pi^2} [4(-1)^n - 4] - \frac{2}{n\pi} [1 - (-1)^n]$$

$$b_{2n+1} = -\frac{8}{(2n+1)\pi^2} + \frac{4}{(2n+1)\pi} = \frac{4}{(2n+1)\pi} \left(1 - \frac{2}{\pi}\right)$$

$$g(x) = \frac{1}{2} - \sum_{n=0}^{\infty} \frac{4}{(2n+1)^2 \pi^2} \cos[(2n+1)x] + \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \left(1 - \frac{2}{\pi}\right) \sin[(2n+1)x]$$

(a)

$$g(0) = g(2\pi) = 0 = \frac{1}{2} - \sum_{n=0}^{\infty} \frac{4}{(2n+1)^2\pi^2}$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

(b)

$$\left[\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}\right]^2 = \frac{\pi^4}{64}$$

$$\begin{aligned}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\cdots\right)&\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\cdots\right)\\ &= 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\cdots+\frac{1}{3^2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\cdots\right)\\ &+\frac{1}{5^2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\cdots\right)+\frac{1}{7^2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\cdots\right)\\ &+\frac{1}{9^2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\cdots\right) = s + \sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} = \frac{\pi^4}{64}\end{aligned}$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{64} - s$$

$$s = 2\left[\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\cdots+\frac{1}{3^2}\left(\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\cdots\right)+\cdots\right]$$

$$s \approx 0.50732667704348711$$

$$\begin{aligned}\frac{\pi^4}{64} - s &\approx 1.5220170474062880818193801982 61 - 0.50732667704348711\\ &= 1.01469037036280097181938019 8261\end{aligned}$$

$$\frac{\pi^4}{\delta} \approx 1.0146903703628009718193801982 61$$

$$\delta \approx 95.998832628296223472948144984649$$

$$\delta = 96$$

(1)

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} + \sum_{n=0}^{\infty} \frac{1}{(2n+2)^4} = \sum_{n=0}^{\infty} \frac{1}{n^4}$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+2)^4}=\frac{1}{16}\sum_{n=0}^{\infty}\frac{1}{(n+1)^4}$$

$$\frac{1}{16}\int_0^{\infty}\frac{dx}{(x+1)^4}=\frac{1}{16}\int_1^{\infty}\frac{dy}{y^4}=\frac{1}{48}$$

$$\sum_{n=0}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{96}+t$$

$$t=\frac{1}{16}\sum_{n=0}^{\infty}\frac{1}{(n+1)^4}\approx 0.067645202106928815$$

(2)

$$\frac{\pi^4}{96}+t=\frac{\pi^4}{90}$$

$$\begin{aligned}t=\frac{\pi^4}{90}-\frac{\pi^4}{96}&\approx 1.0823232337111381915160036965412\\&-1.0146780316041920545462534655073\\&\approx 0.0676452021069461369697502310338 2\end{aligned}$$

$$\frac{\pi^4}{96}\times\frac{16}{15}=\frac{\pi^4}{6}\times\frac{16}{15}=\frac{\pi^4}{90}$$

$$t=\frac{1}{16}\sum_{n=0}^{\infty}\frac{1}{(n+1)^4}=\frac{1}{16}\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\right)=\frac{1}{16}\sum_{m=1}^{\infty}\frac{1}{m^4}=\frac{\zeta(4)}{16}=\frac{6\pi^4}{8640}=\frac{\pi^4}{1440}=\frac{1}{16}\times\frac{\pi^4}{90}$$

$$\therefore \zeta(4)=\frac{\pi^4}{90}$$

36.4 Evaluate the following integral:

$$\int_0^{\infty}\frac{x^3dx}{e^x-1}=\int_0^{\infty}\frac{x^3e^{-x}dx}{1-e^{-x}}$$

$$\frac{1}{1-e^{-x}}=1+e^{-x}+e^{-2x}+e^{-3x}+\cdots=\sum_{n=0}^{\infty}(e^{-x})^n=\sum_{n=0}^{\infty}e^{-nx}$$

$$\int_0^{\infty}\frac{x^3e^{-x}dx}{1-e^{-x}}=\sum_{n=0}^{\infty}\int_0^{\infty}x^3e^{-(n+1)x}dx$$

$$u(x)=x^3$$

$$dv=e^{-(n+1)x}dx$$

$$v(x)=\int e^{-(n+1)x}dx=-\frac{e^{-(n+1)x}}{n+1}$$

$$\int_0^\infty x^3 e^{-(n+1)x} dx = \frac{3}{n+1}\int_0^\infty x^2 e^{-(n+1)x} dx$$

$$u(x) = x^2$$

$$dv = e^{-(n+1)x} dx$$

$$v(x) = \int e^{-(n+1)x} dx = -\frac{e^{-(n+1)x}}{n+1}$$

$$\int_0^\infty x^2 e^{-(n+1)x} dx = \frac{2}{n+1}\int_0^\infty x\, e^{-(n+1)x} dx$$

$$u(x) = x$$

$$dv = e^{-(n+1)x} dx$$

$$v(x) = \int e^{-(n+1)x} dx = -\frac{e^{-(n+1)x}}{n+1}$$

$$\int_0^\infty x\, e^{-(n+1)x} dx = \frac{1}{n+1}\int_0^\infty e^{-(n+1)x} dx = \frac{1}{(n+1)^2}$$

$$\int_0^\infty \frac{x^3 e^{-x} dx}{1-e^{-x}} = 6\sum_{n=0}^{\infty}\frac{1}{(n+1)^4} = 6\zeta(4) = \frac{6\pi^4}{90} = \frac{\pi^4}{15}$$

36.5

$$y(x,t) = \frac{8h}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\left[\frac{(2n-1)\pi x}{2}\right]\cos\left[\frac{(2n-1)at}{2}\right]$$

$$y\left(1,\frac{2\pi}{a}\right) = \frac{8h}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\left[\frac{(2n-1)\pi}{2}\right]\cos[(2n-1)\pi] = \frac{8h}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-1)^{n+1}(-1)^{n+1}}{(2n-1)^2}$$

$$= \frac{8h}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{3n+3}}{(2n-1)^2} = \frac{8h}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2} = \frac{8h}{\pi^2}\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\cdots\right) = \frac{8h}{\pi^2}\times\frac{\pi}{4} = \frac{2h}{\pi}$$

$$y(1,0) = \frac{8h}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\left[\frac{(2n-1)\pi}{2}\right] = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-1)^{n+1}}{(2n-1)^2}$$

$$= \frac{8h}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{2n+2}}{(2n-1)^2} = \frac{8h}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{8h}{\pi^2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right) =$$

$$\frac{A_1}{A_0} = 1, \frac{A_2}{A_0} = 0, \frac{A_3}{A_0} = \frac{1}{3^2} = \frac{1}{9}$$

36.6

$$h(x) = \frac{x}{2\pi} \ \forall x \in [0,2\pi)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} h(x)\cos nx\, dx = \frac{1}{2\pi^2}\int_0^{2\pi} x\cos nx\, dx = \frac{x}{n}\sin nx\Big]_{x=0}^{x=2\pi} - \frac{1}{n}\int_0^{2\pi} \sin nx\, dx = \frac{1}{n^2}\cos nx]_{x=0}^{x=2\pi}$$
$$= 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} h(x)\sin nx\, dx = \frac{1}{2\pi^2}\int_0^{2\pi} x\sin nx\, dx$$

$$\int_0^{2\pi} x\sin nx\, dx = -\frac{x}{n}\cos nx\Big]_{x=0}^{x=2\pi} - \frac{1}{n}\int_0^{2\pi} \cos nx\, dx = -\frac{2\pi}{n}$$

$$b_n = -\frac{1}{\pi n}$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} h(x)\, dx = \frac{1}{2\pi^2}\int_0^{2\pi} x\, dx = \frac{1}{2\pi^2}\times\frac{1}{2}x^2]_{x=0}^{x=2\pi} = 1$$

$$h(x) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin nx}{n}$$

Conjecture:

$$h\left(\frac{\pi}{2}\right) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n} = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)} = \frac{1}{2} - \frac{1}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right) = 0.25 = \frac{1}{4}$$

$$\therefore \frac{\pi}{2} - \frac{\pi}{4} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)} = \frac{\pi}{4} = \tan^{-1} 1$$

36.7

$$S\frac{\partial^2 y}{\partial x^2} - \sigma\frac{\partial^2 y}{\partial t^2} = 0$$

$$S \equiv tension\ and\ \sigma \equiv mass\ density$$

$$y(x,0) = \begin{cases} A_0\frac{x}{x_p} \ \forall x \in [0, x_p] \\ A_0\left(1 - \frac{x - x_p}{L - x_p}\right) \ \forall x \in (x_p, L] \end{cases}$$

$$\dot{y}(x,0) = 0$$

$$A_0 \equiv amplitude, L \equiv the\ length\ of\ the\ string$$

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[a_n \sin\left(\frac{n\pi ct}{L}\right) + b_n \cos\left(\frac{n\pi ct}{L}\right)\right]$$

$$\dot{y}(x,t) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right)\left[a_n \cos\left(\frac{n\pi ct}{L}\right) - b_n \sin\left(\frac{n\pi ct}{L}\right)\right]$$

$$c = \sqrt{\frac{S}{\sigma}}$$

$$a_n = 0$$

$$b_n = A_0 \frac{2L^2 \sin\left(\frac{n\pi x_p}{L}\right)}{n^2\pi^2 x_p(L - x_p)}$$

$$y(x,t) = 2A_0 \sum_{n=1}^{\infty} \frac{L^2 \sin\left(\frac{n\pi x_p}{L}\right)}{n^2\pi^2 x_p(L - x_p)} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right)$$

$$y(x,0) = 2A_0 \sum_{n=1}^{\infty} \frac{L^2 \sin\left(\frac{n\pi x_p}{L}\right)}{n^2\pi^2 x_p(L - x_p)} \sin\left(\frac{n\pi x}{L}\right)$$

$$T = \frac{2L}{c} = 2L\sqrt{\frac{\sigma}{S}}$$

$$y(x,t) = 8A_0 \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right)$$

$$y(x,0) = \frac{8A_0}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi x}{L}\right) = \frac{8A_0}{\pi^2}\left[\sin\left(\frac{\pi x}{L}\right) + \frac{1}{4}\sin\left(\frac{2\pi x}{L}\right) + \frac{1}{9}\sin\left(\frac{3\pi x}{L}\right) + \cdots\right]$$

$$\frac{T}{2} = \frac{L}{c}$$

$$y\left(x, \frac{L}{c}\right) = 8A_0 \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} \sin\left(\frac{n\pi x}{L}\right) \cos(n\pi) = \frac{8A_0}{\pi^2}\left[-\sin\left(\frac{\pi x}{L}\right) + \frac{1}{4}\sin\left(\frac{2\pi x}{L}\right) - \frac{1}{9}\sin\left(\frac{3\pi x}{L}\right) + \cdots\right]$$

$$y\left(\frac{L}{2}, \frac{L}{c}\right) = -\frac{8A_0}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

36.8

$$f(x) = \sin x \; \forall x \in [0, \pi)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(2nx) + b_n \sin(2nx)]$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin x \, dx = -\frac{2}{\pi}\cos x\}_{x=0}^{x=\pi} = -\frac{2}{\pi}(-1-1) = \frac{4}{\pi} \approx 1.2732395447351626861510701069801$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos(2nx) \, dx = \frac{2}{\pi}\left\{\int_0^{\pi} \sin[(1+2n)x] \, dx - \int_0^{\pi} \sin[(1-n)x] \, dx\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{\cos[(1+2n)x]}{1+2n}\right]_{x=0}^{x=\pi} - \frac{\cos[(1-2n)x]}{1-2n}\Bigg]_{x=0}^{x=\pi}\Bigg\} =$$

$$= \frac{1}{\pi}\left\{-\frac{\cos[(1+2n)x]}{1+2n}\right]_{x=0}^{x=\pi} + \frac{\cos[(1-2n)x]}{2n-1}\Bigg]_{x=0}^{x=\pi}\Bigg\}$$

$$a_1 = \frac{1}{\pi}\left[-\frac{\cos(3\pi)}{3} + \frac{1}{3} + \cos\pi - 1\right] = \frac{1}{\pi}\left(\frac{2}{3} - 2\right) = \frac{1}{\pi}\left(\frac{2}{3} - \frac{6}{3}\right) = -\frac{4}{3\pi}$$

$$\approx -0.42441318157838756205035670232 67$$

$$a_2 = \frac{1}{\pi}\left[-\frac{\cos(5\pi)}{5} + \frac{1}{5} + \frac{\cos(3\pi)}{3} - \frac{1}{3}\right] = \frac{2}{\pi}\left(\frac{1}{5} - \frac{1}{3}\right) = \frac{2}{\pi}\left(\frac{3}{15} - \frac{5}{15}\right) = -\frac{4}{15\pi}$$

$$\approx -0.084882636315677512410071340465 34$$

$$a_2 = \frac{1}{\pi}\left[-\frac{\cos(7\pi)}{7} + \frac{1}{7} + \frac{\cos(5\pi)}{5} - \frac{1}{5}\right] = \frac{2}{\pi}\left(\frac{1}{7} - \frac{1}{5}\right) = \frac{2}{\pi}\left(\frac{5}{35} - \frac{7}{35}\right) = -\frac{4}{35\pi}$$

$$\approx -0.036378272706718933890030574485 15$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin x \sin(2nx) \, dx$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\int_0^{\pi} \sin x \sin(2nx) \, dx = \int_0^{\pi} x \sin(2nx) \, dx - \frac{1}{3!}\int_0^{\pi} x^3 \sin(2nx) \, dx + \frac{1}{5!}\int_0^{\pi} x^5 \sin(2nx) \, dx - \cdots$$

$$u(x) = x$$

$$dv = \sin(2nx) \, dx$$

$$v(x) = -\frac{1}{2n}\cos(2nx)$$

$$\int_0^{\pi} x \sin(2nx) \, dx = -\frac{x}{2n}\cos(2nx)\Big]_{x=0}^{x=\pi} + \frac{1}{2n}\int_0^{\pi} \cos(2nx) \, dx = \frac{1}{n^2}\sin(2nx)\Big]_{x=0}^{x=\pi} = 0$$

$$u(x) = x^3$$

$$dv = \sin(2nx) \, dx$$

$$v(x) = -\frac{1}{2n}\cos(2nx)$$

$$\int_0^\pi x^3 \sin(2nx)\,dx = -\frac{x^3}{2n}\cos(2nx)\Big]_{x=0}^{x=\pi} + \frac{3}{2n}\int_0^\pi x^2\cos(2nx)\,dx$$

$$u(x) = x^2$$

$$dv = \cos(2nx)\,dx$$

$$v(x) = \frac{1}{2n}\sin(2nx)$$

$$\int_0^\pi x^2\cos(2nx)\,dx = \frac{x^2}{2n}\sin(2nx)\Big]_{x=0}^{x=\pi} - \frac{1}{n}\int_0^\pi x\sin(2nx)\,dx = 0$$

Conjecture:

$$\int_0^\pi x^{2n+1}\sin(2nx)\,dx = 0 \therefore b_n = 0$$

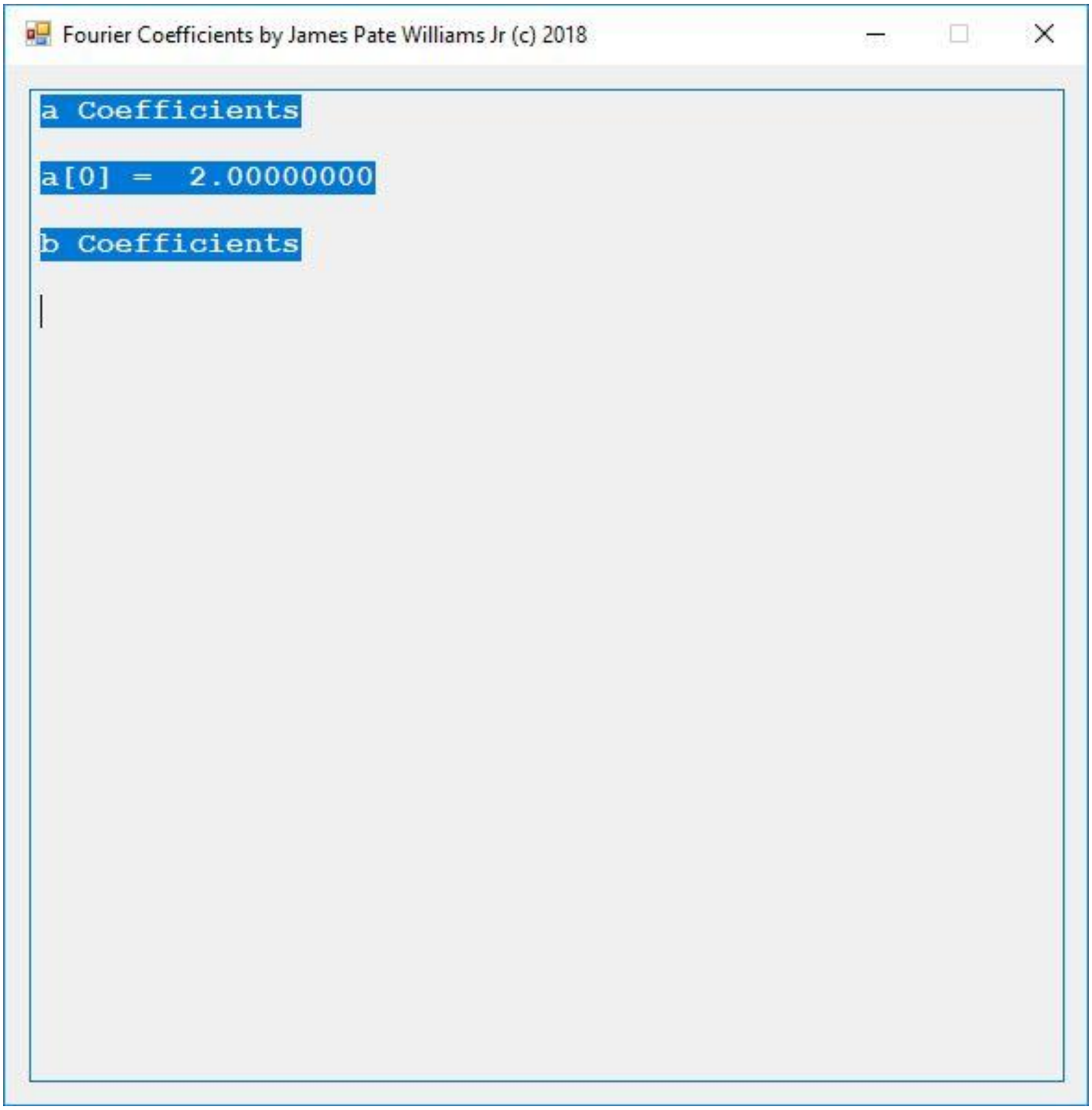
(a)

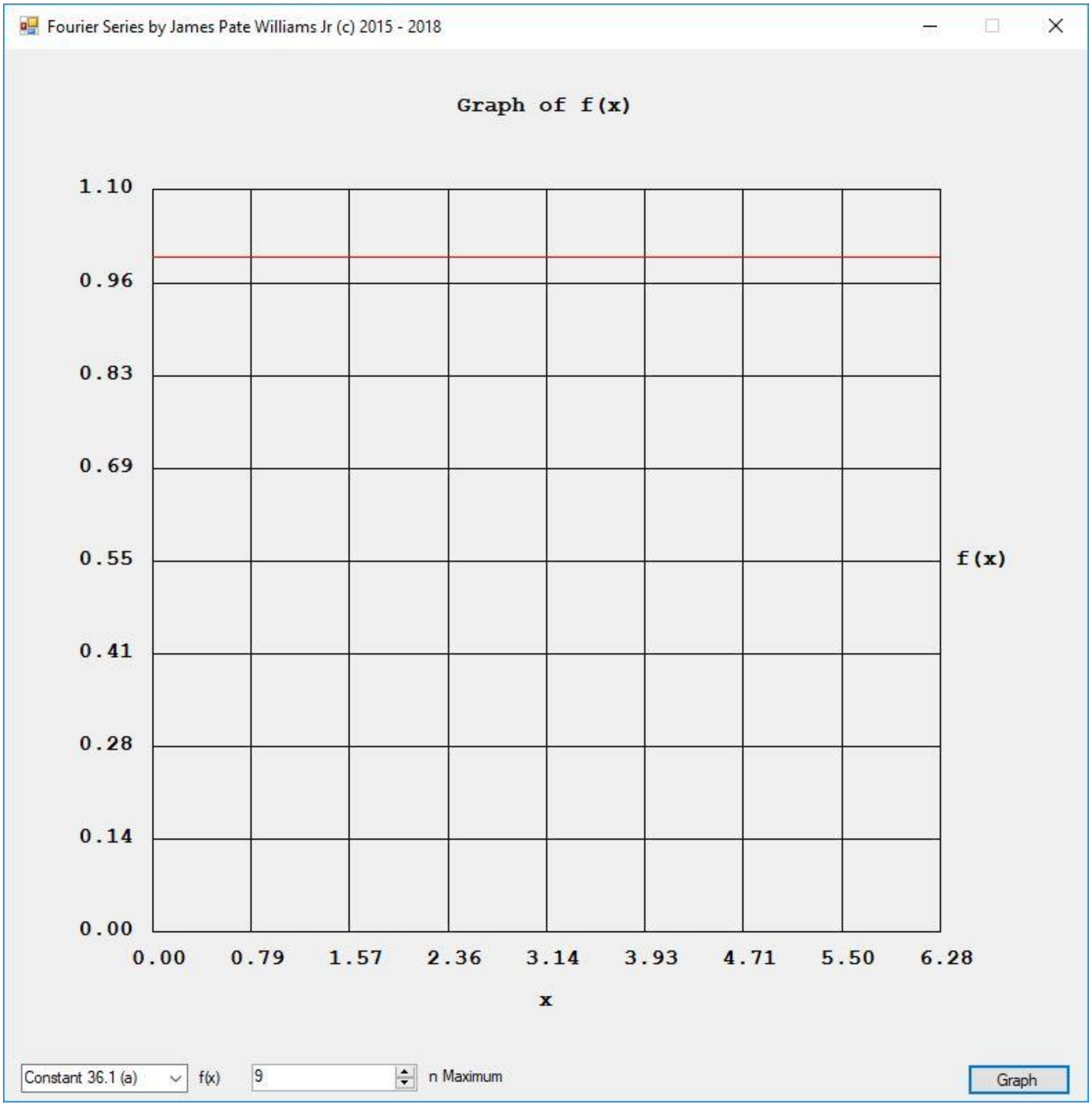
$$\frac{1}{\pi}\int_0^\pi |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 = \frac{8}{\pi^2} + \frac{16}{\pi^2}\left(\frac{1}{9} + \frac{1}{225} + \frac{1}{1225} + \cdots\right) = \frac{2a_0}{\pi} + a_0^2\left(\frac{1}{9} + \frac{1}{225} + \frac{1}{1225} + \cdots\right)$$

(b)

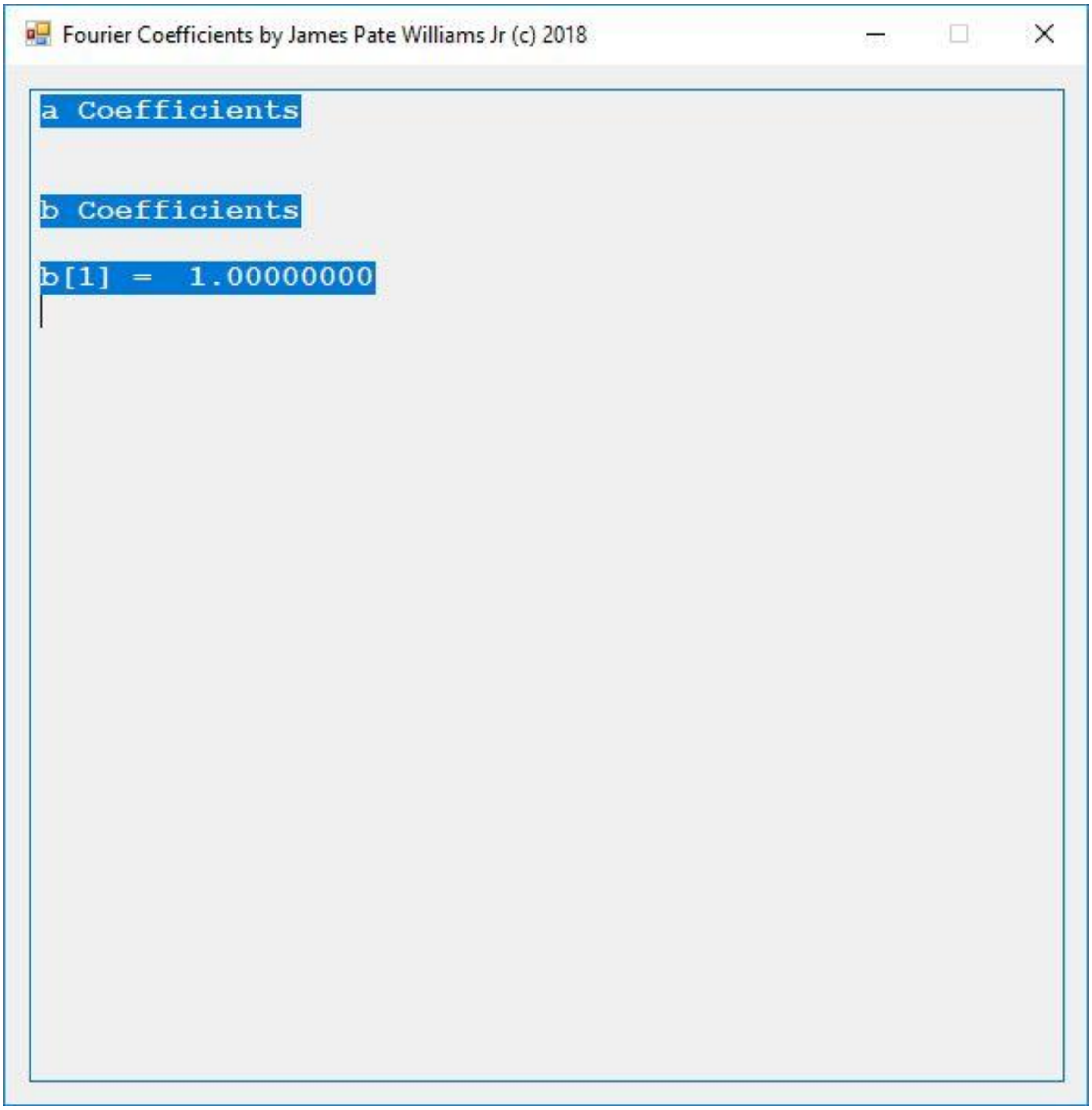
$$a_2 = -\frac{4}{15\pi} = -\frac{a_0}{15}$$

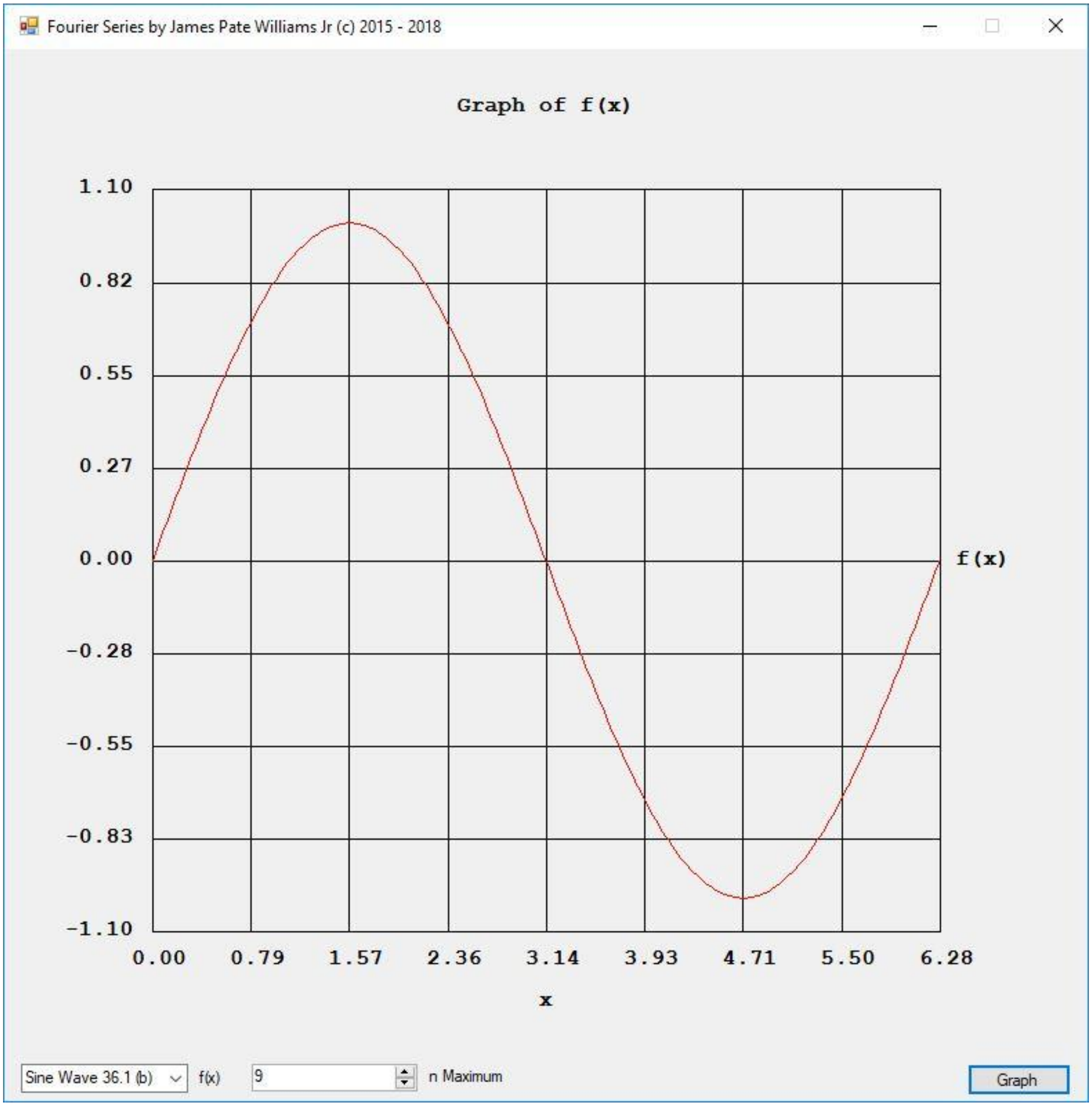
Fourier coefficients and graph for Exercise 36.1 (a) for f(x) = 1 for all x in the half-open interval [x0, P).


Fourier Coefficients by James Pate Williams Jr (c) 2018
a Coefficients
a[0] = 2.00000000
b Coefficients


Fourier Series by James Pate Williams Jr (c) 2015 - 2018
Graph of f(x)
1.10
0.96
0.83
0.69
0.55
0.41
0.28
0.14
0.00
0.00
0.79
1.57
2.36
3.14
3.93
4.71
5.50
6.28
x
f(x)
Constant 36.1 (a)
f(x)
9
n Maximum
Graph

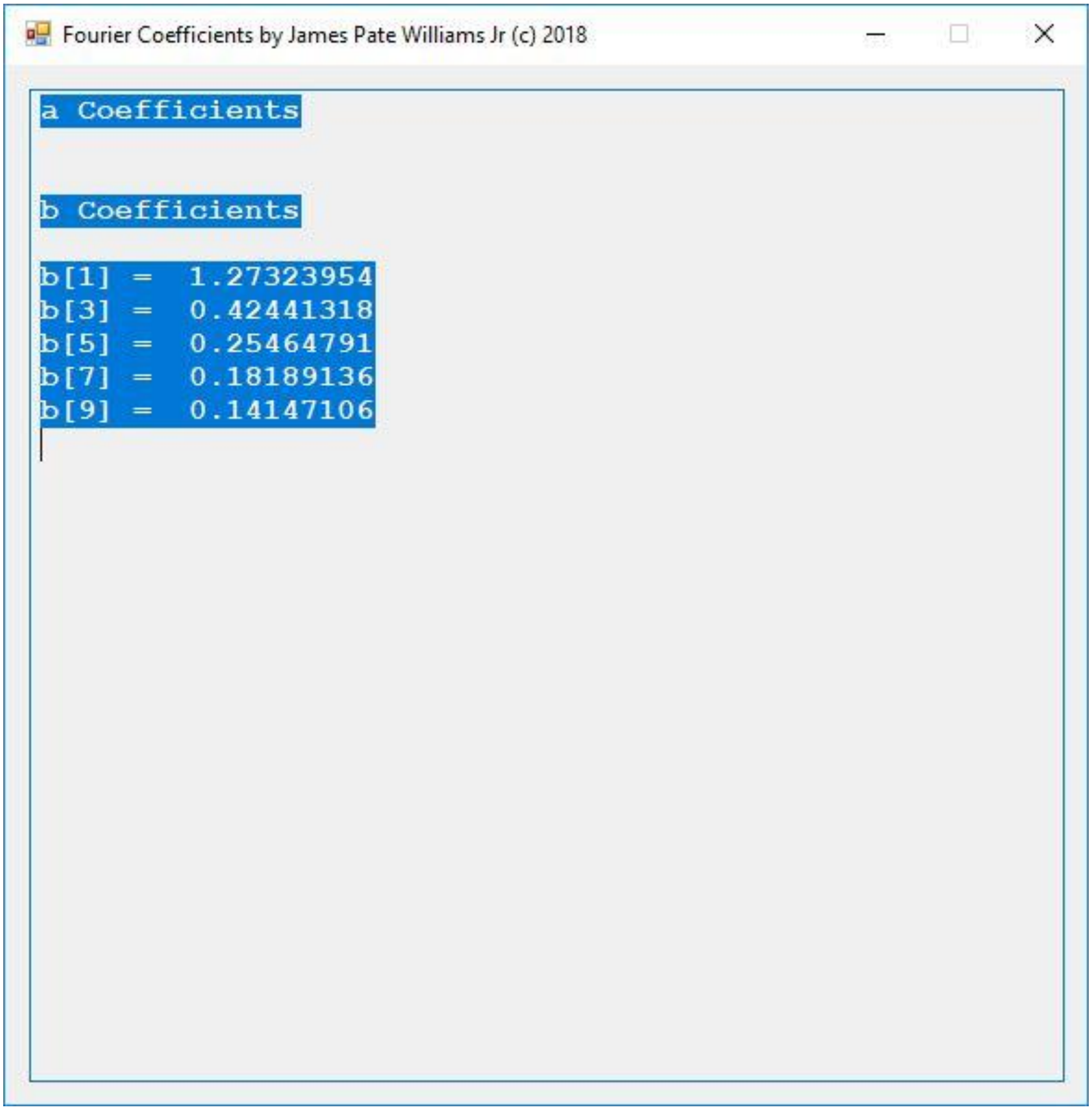
Fourier coefficients and graph for Exercise 36.1 (b) f(x) = sin x for all x in the half-open interval [0, 2π).

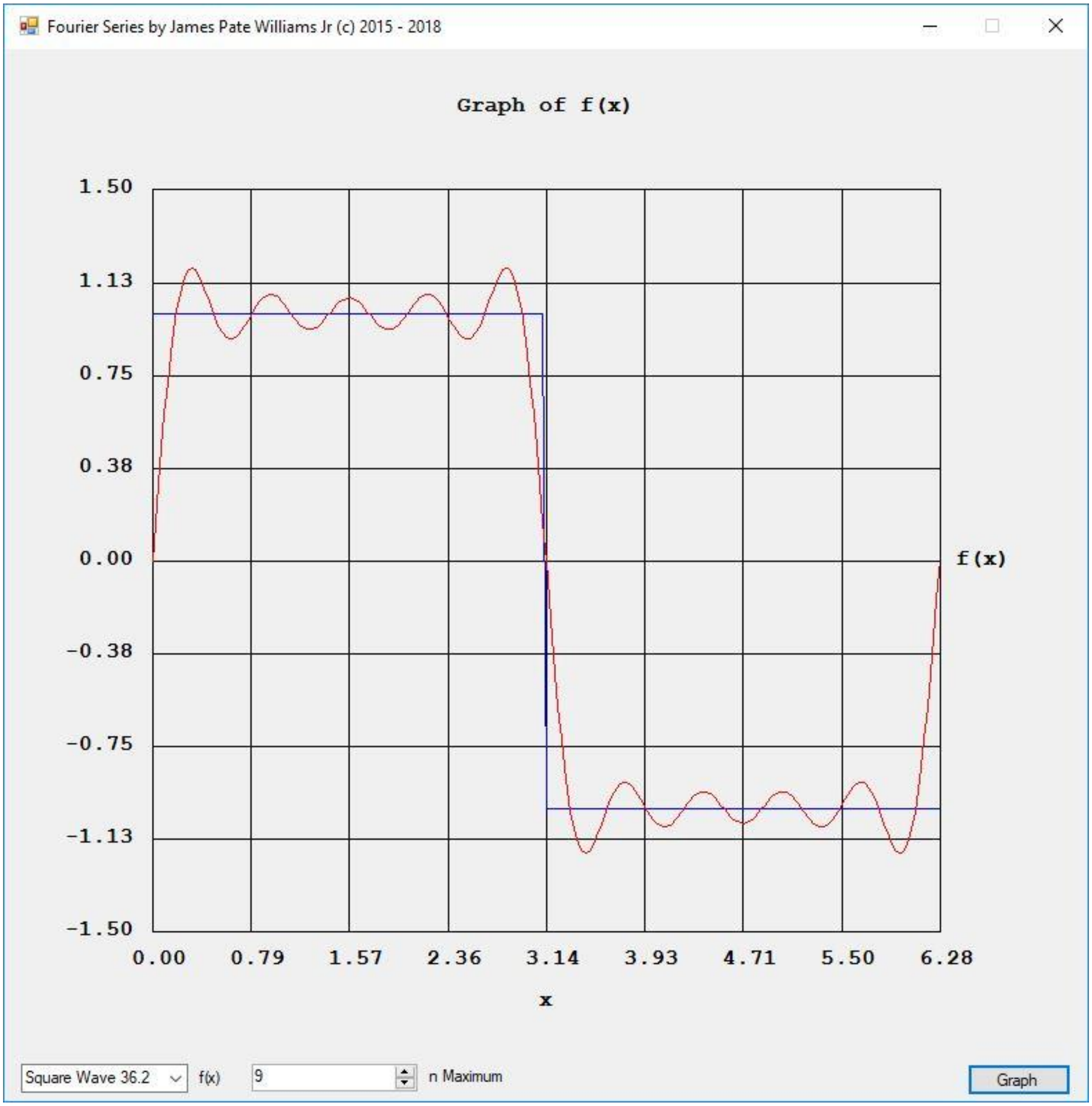




Fourier Series by James Pate Williams Jr (c) 2015 - 2018
Graph of f(x)
1.10
0.82
0.55
0.27
0.00
-0.28
-0.55
-0.83
-1.10
f(x)
0.00
0.79
1.57
2.36
3.14
3.93
4.71
5.50
6.28
x
Sine Wave 36.1 (b)
f(x)
9
n Maximum
Graph

Fourier coefficients and graph for Exercise 36.2, the square wave.




Fourier Series by James Pate Williams Jr (c) 2015 - 2018
Graph of f(x)
1.50
1.13
0.75
0.38
0.00
-0.38
-0.75
-1.13
-1.50
f(x)
0.00
0.79
1.57
2.36
3.14
3.93
4.71
5.50
6.28
x
Square Wave 36.2
f(x)
9
n Maximum
Graph

Infinite Series by James Pate Williams Jr (c) 2018

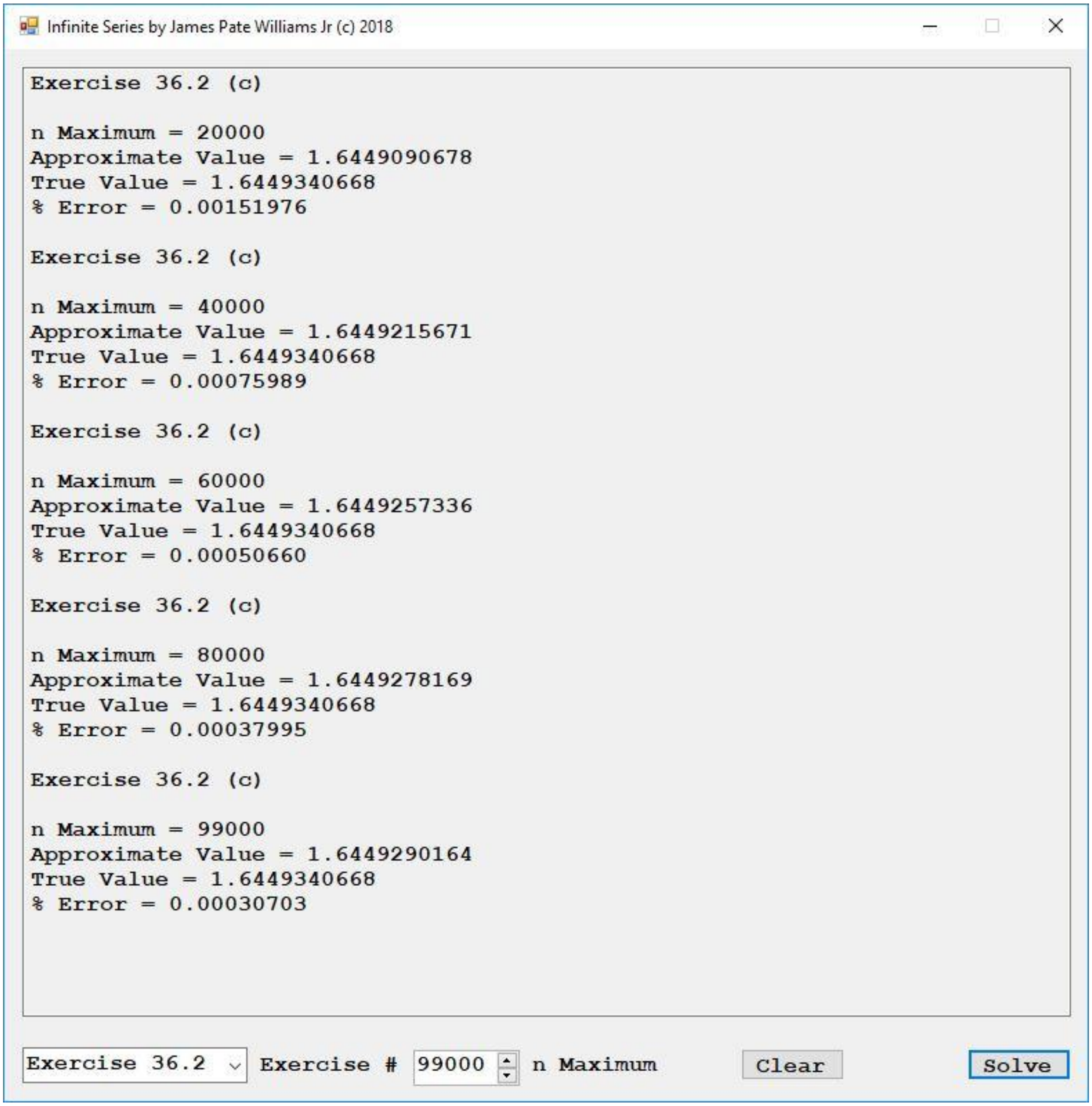
```
Exercise 36.2 (a)

n Maximum = 20000
Approximate Value = 0.7854106628
True Value = 0.7853981634
% Error = 0.00159147

Exercise 36.2 (a)

n Maximum = 40000
Approximate Value = 0.7854044132
True Value = 0.7853981634
% Error = 0.00079575

Exercise 36.2 (a)

n Maximum = 60000
Approximate Value = 0.7854023300
True Value = 0.7853981634
% Error = 0.00053051

Exercise 36.2 (a)

n Maximum = 80000
Approximate Value = 0.7854012884
True Value = 0.7853981634
% Error = 0.00039788

Exercise 36.2 (a)

n Maximum = 99000
Approximate Value = 0.7854006886
True Value = 0.7853981634
% Error = 0.00032152
```

Exercise 36.2 Exercise # 99000 n Maximum Clear Solve

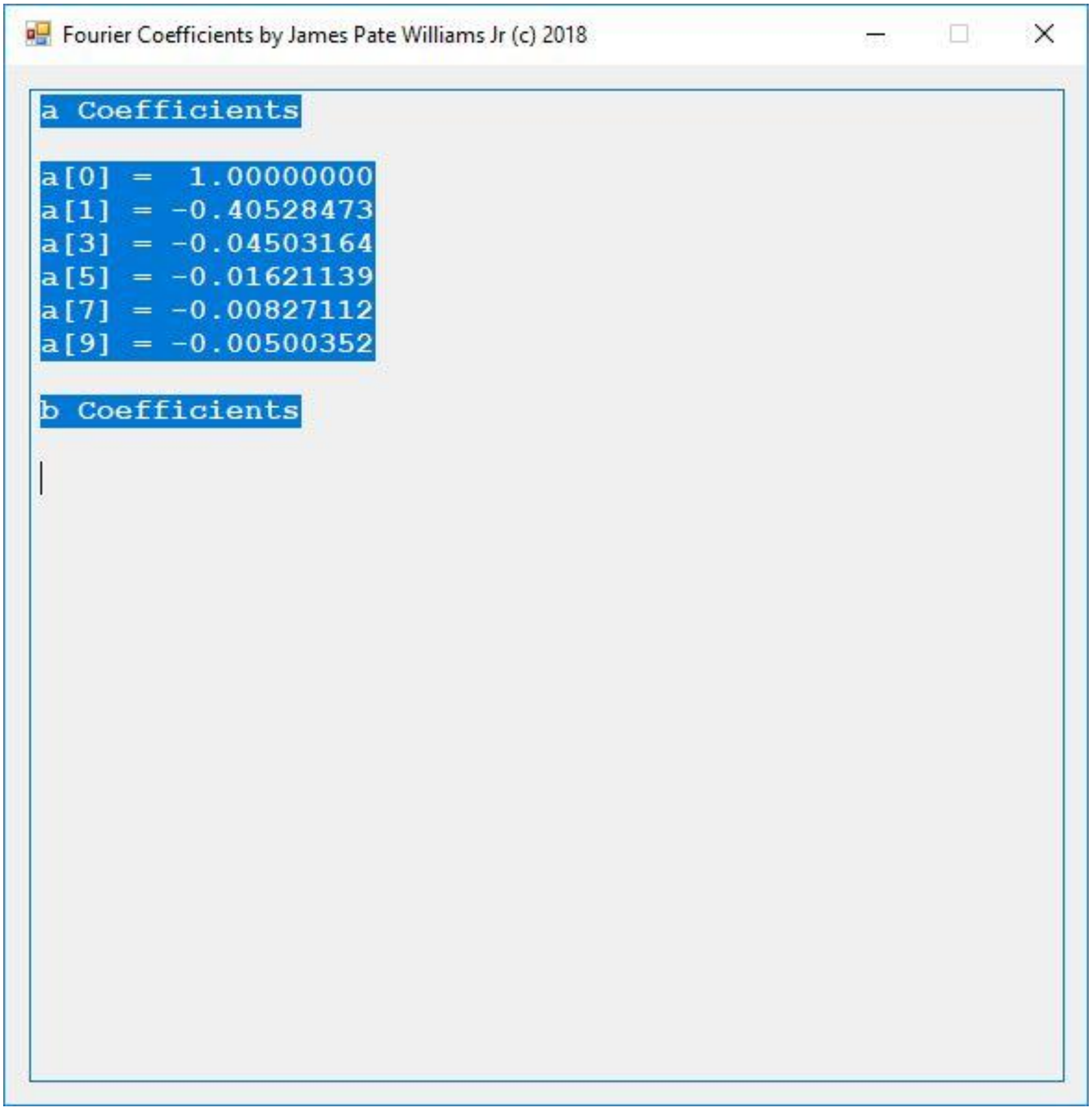
Infinite Series by James Pate Williams Jr (c) 2018

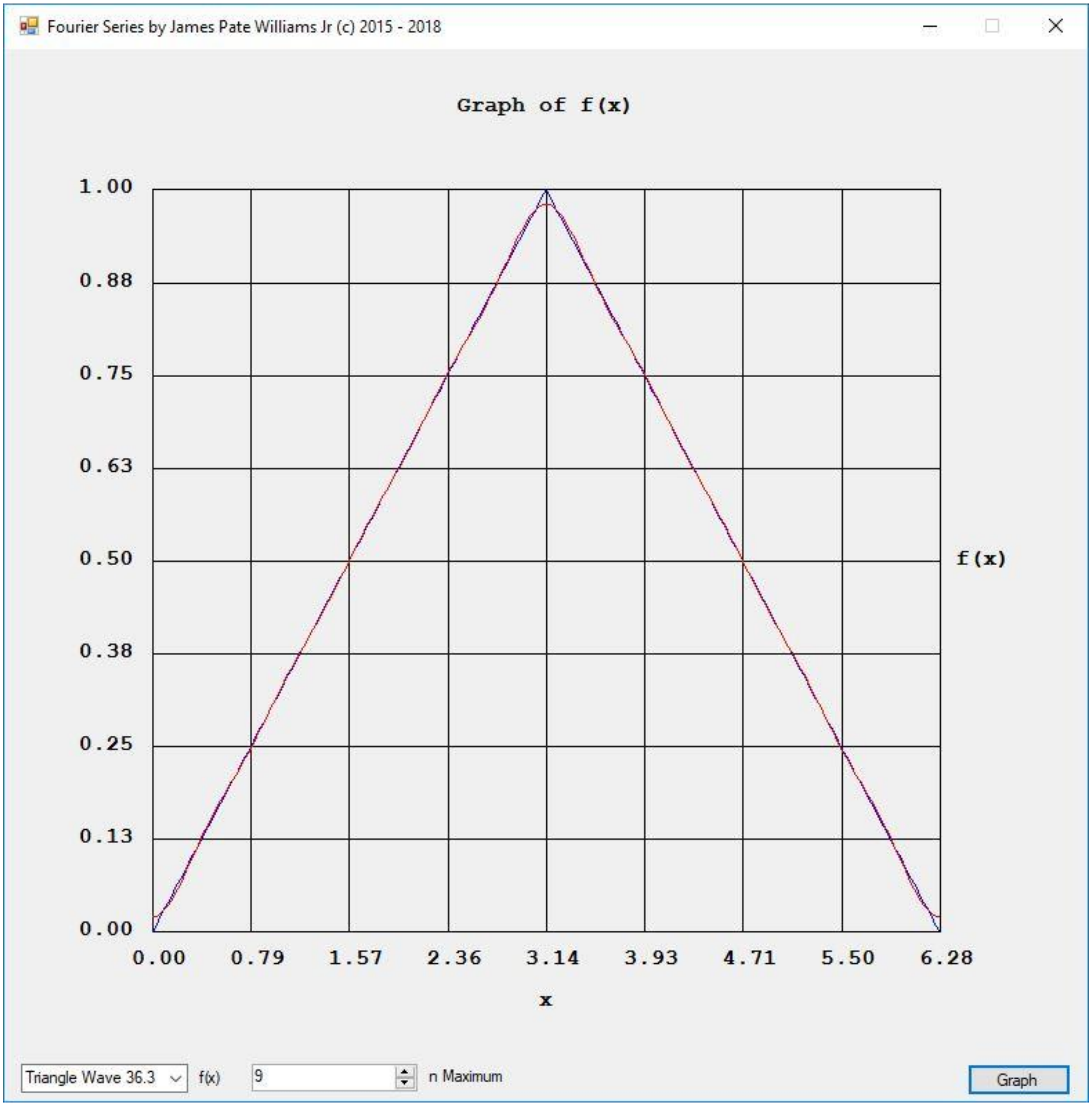
```
Exercise 36.2 (b)

n Maximum = 20000
Approximate Value = 1.2336880508
True Value = 1.2337005501
% Error = 0.00101316

Exercise 36.2 (b)

n Maximum = 60000
Approximate Value = 1.2336963835
True Value = 1.2337005501
% Error = 0.00033773

Exercise 36.2 (b)

n Maximum = 80000
Approximate Value = 1.2336974252
True Value = 1.2337005501
% Error = 0.00025330

Exercise 36.2 (b)

n Maximum = 90000
Approximate Value = 1.2336977724
True Value = 1.2337005501
% Error = 0.00022516

Exercise 36.2 (b)

n Maximum = 99000
Approximate Value = 1.2336980249
True Value = 1.2337005501
% Error = 0.00020469
```

Exercise 36.2 | Exercise # | 99000 | n Maximum | Clear | Solve

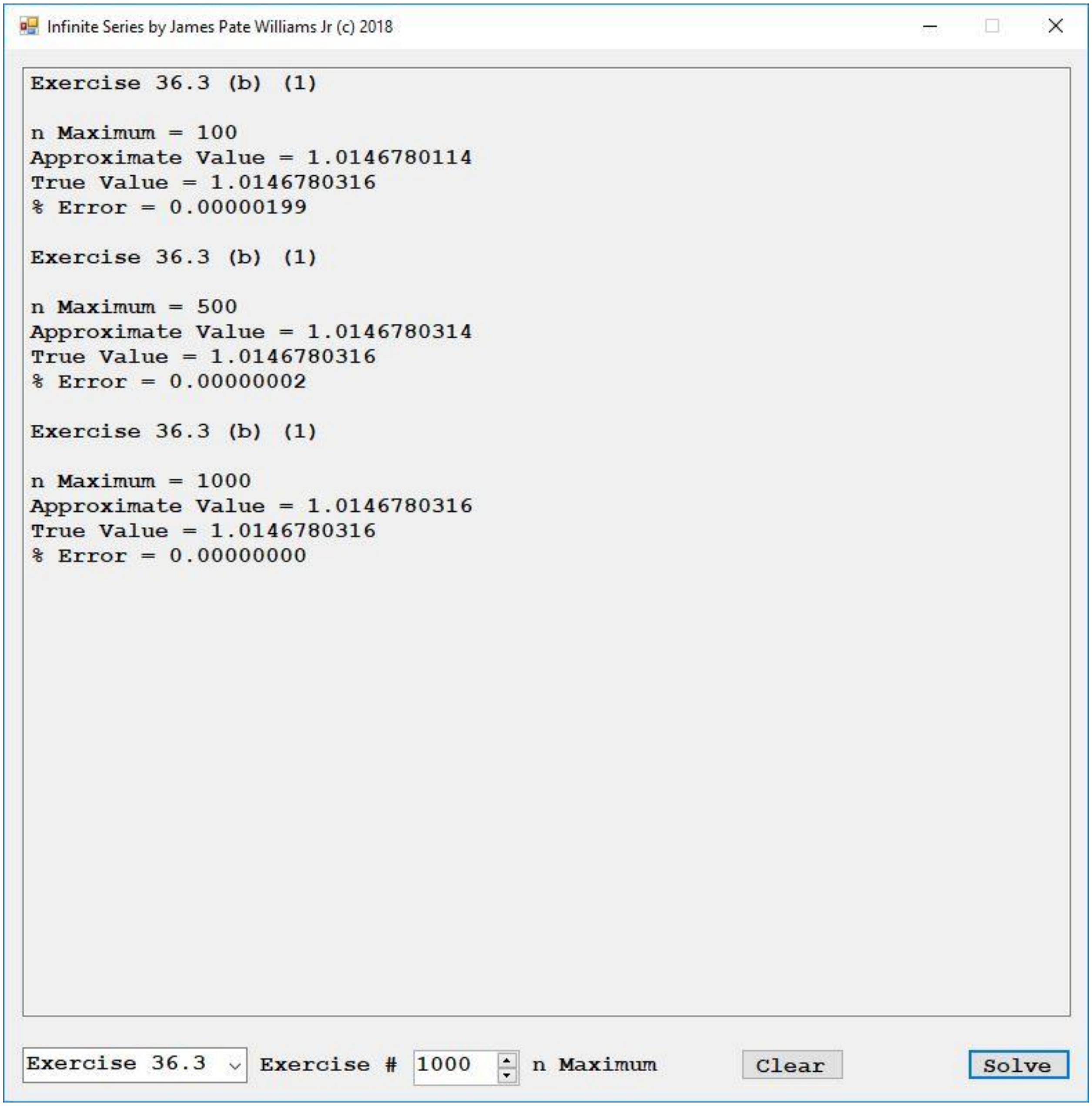

Infinite Series by James Pate Williams Jr (c) 2018
Exercise 36.2 (c)
n Maximum = 20000
Approximate Value = 1.6449090678
True Value = 1.6449340668
% Error = 0.00151976
Exercise 36.2 (c)
n Maximum = 40000
Approximate Value = 1.6449215671
True Value = 1.6449340668
% Error = 0.00075989
Exercise 36.2 (c)
n Maximum = 60000
Approximate Value = 1.6449257336
True Value = 1.6449340668
% Error = 0.00050660
Exercise 36.2 (c)
n Maximum = 80000
Approximate Value = 1.6449278169
True Value = 1.6449340668
% Error = 0.00037995
Exercise 36.2 (c)
n Maximum = 99000
Approximate Value = 1.6449290164
True Value = 1.6449340668
% Error = 0.00030703
Exercise 36.2
Exercise #
99000
n Maximum
Clear
Solve

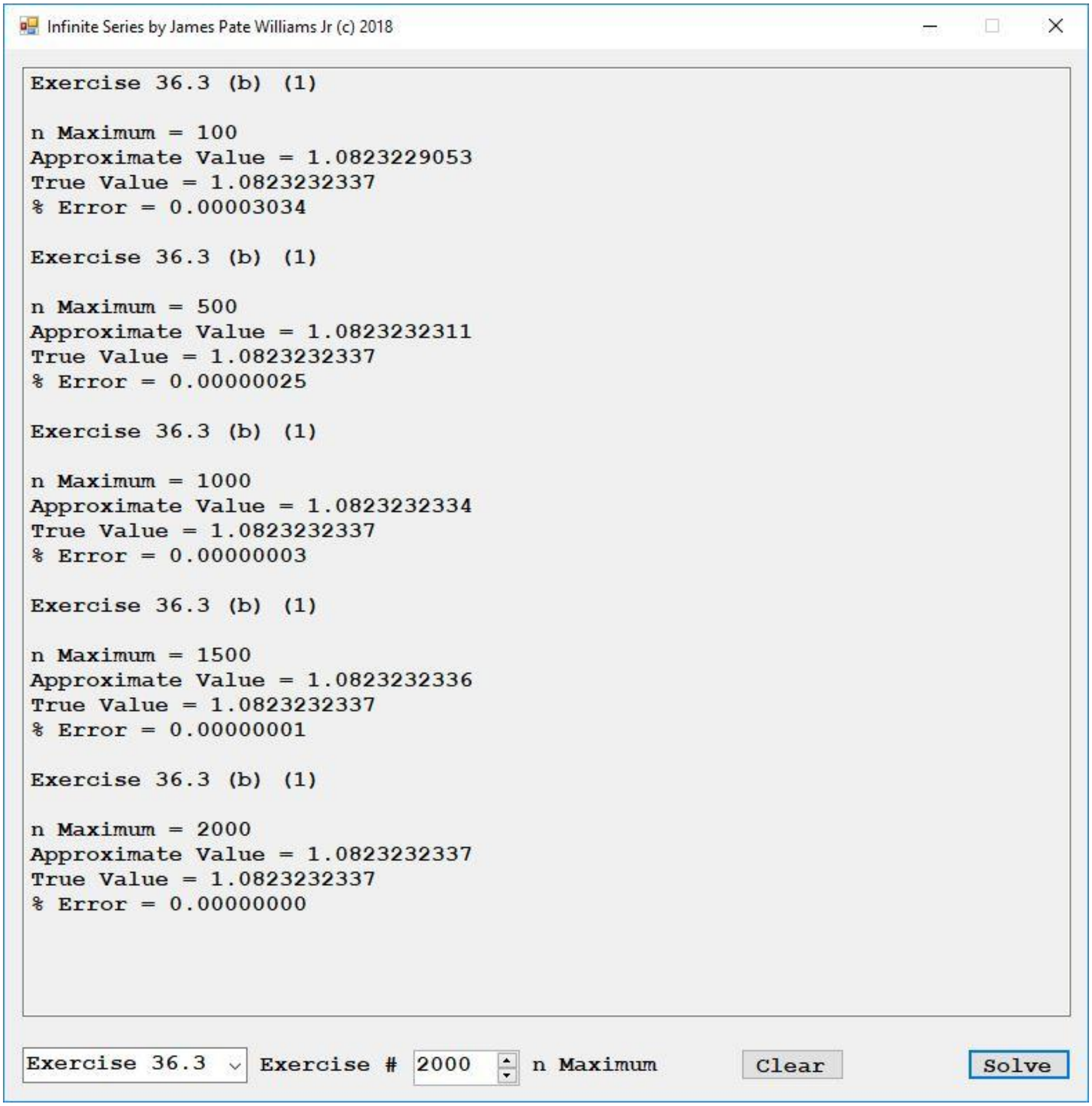
Fourier coefficients and graph for Exercise 36.3, the triangle wave.

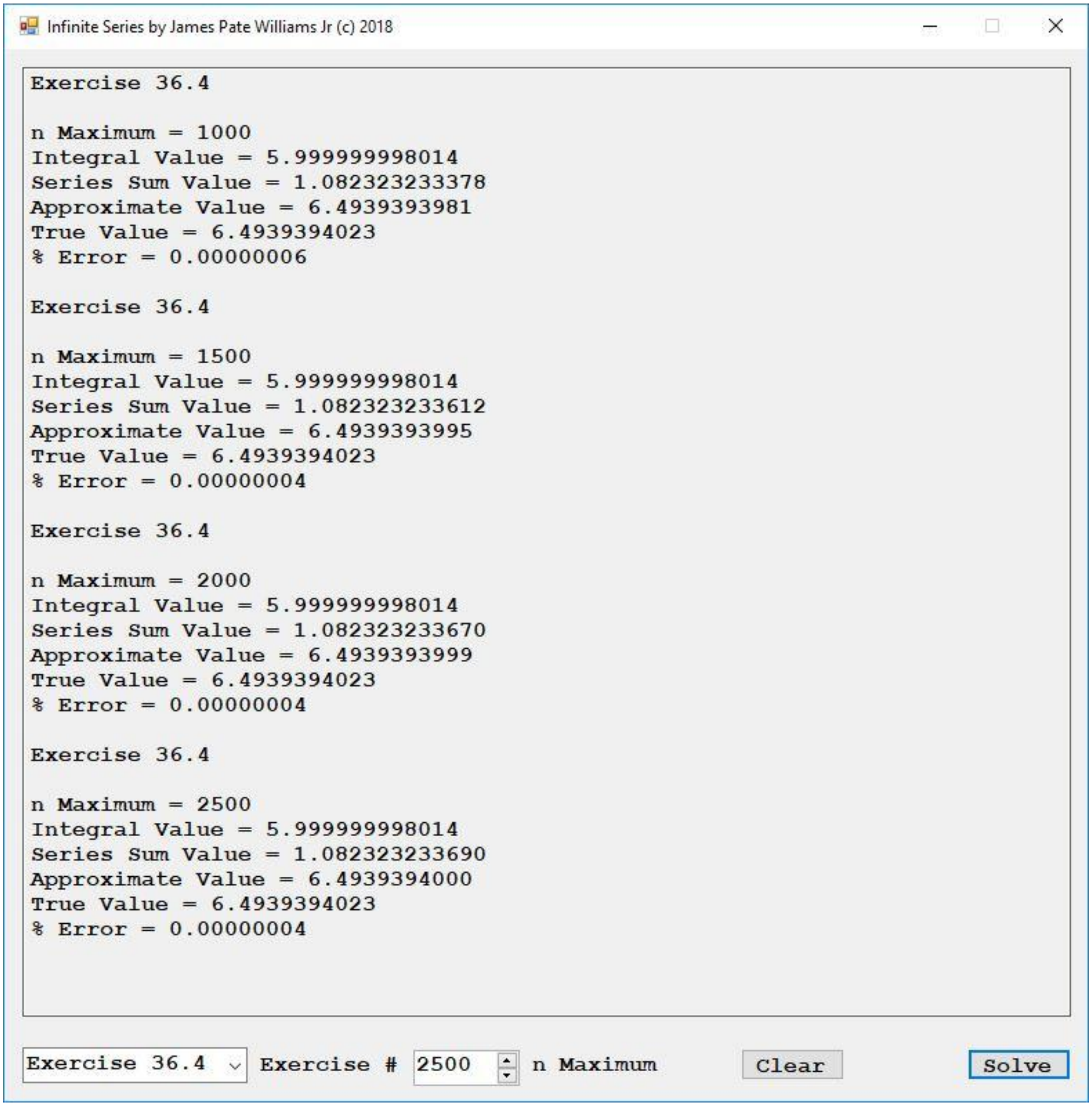




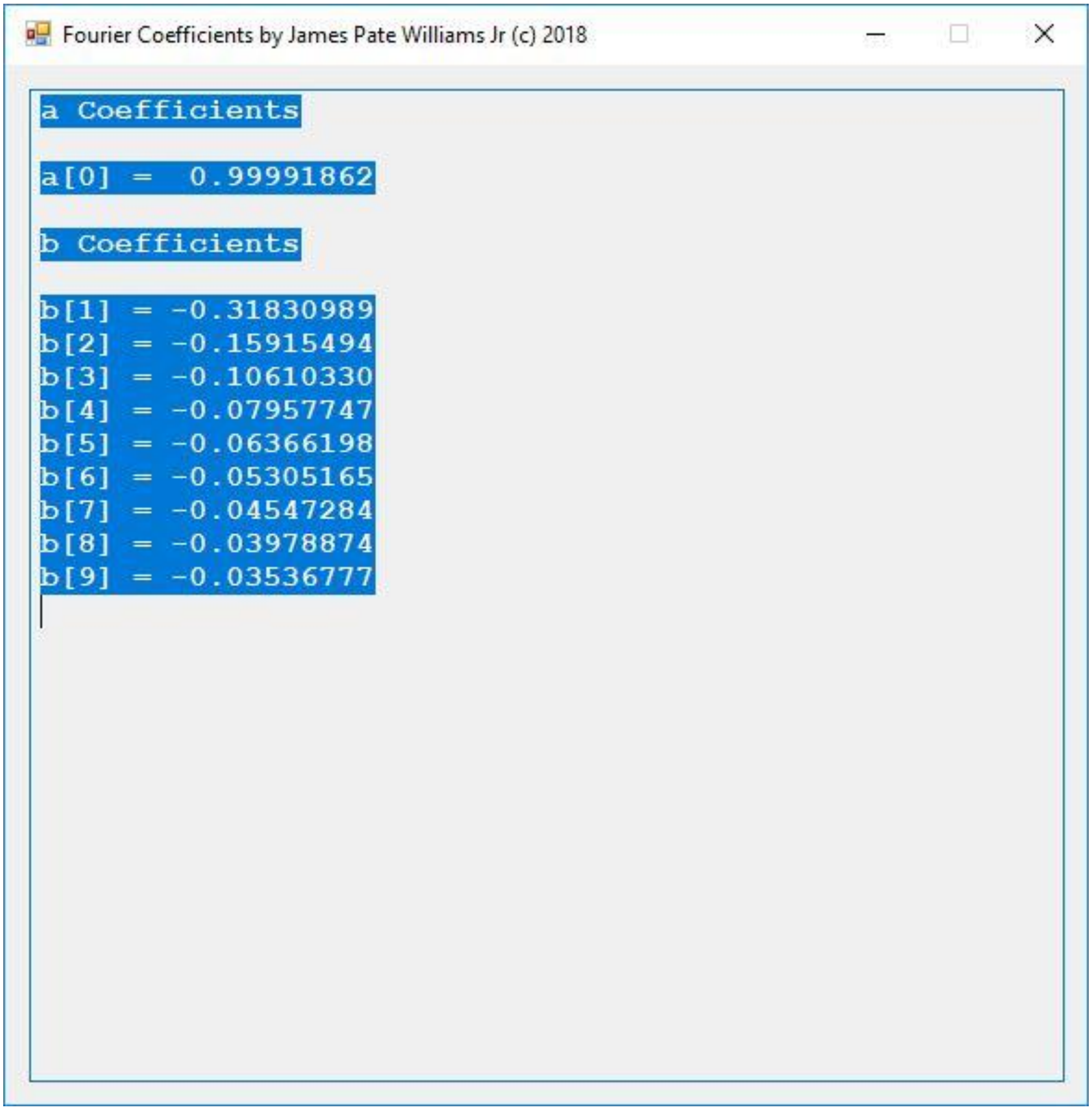
Fourier Series by James Pate Williams Jr (c) 2015 - 2018
Graph of f(x)
1.00
0.88
0.75
0.63
0.50
0.38
0.25
0.13
0.00
0.00
0.79
1.57
2.36
3.14
3.93
4.71
5.50
6.28
x
f(x)
Triangle Wave 36.3
f(x)
9
n Maximum
Graph

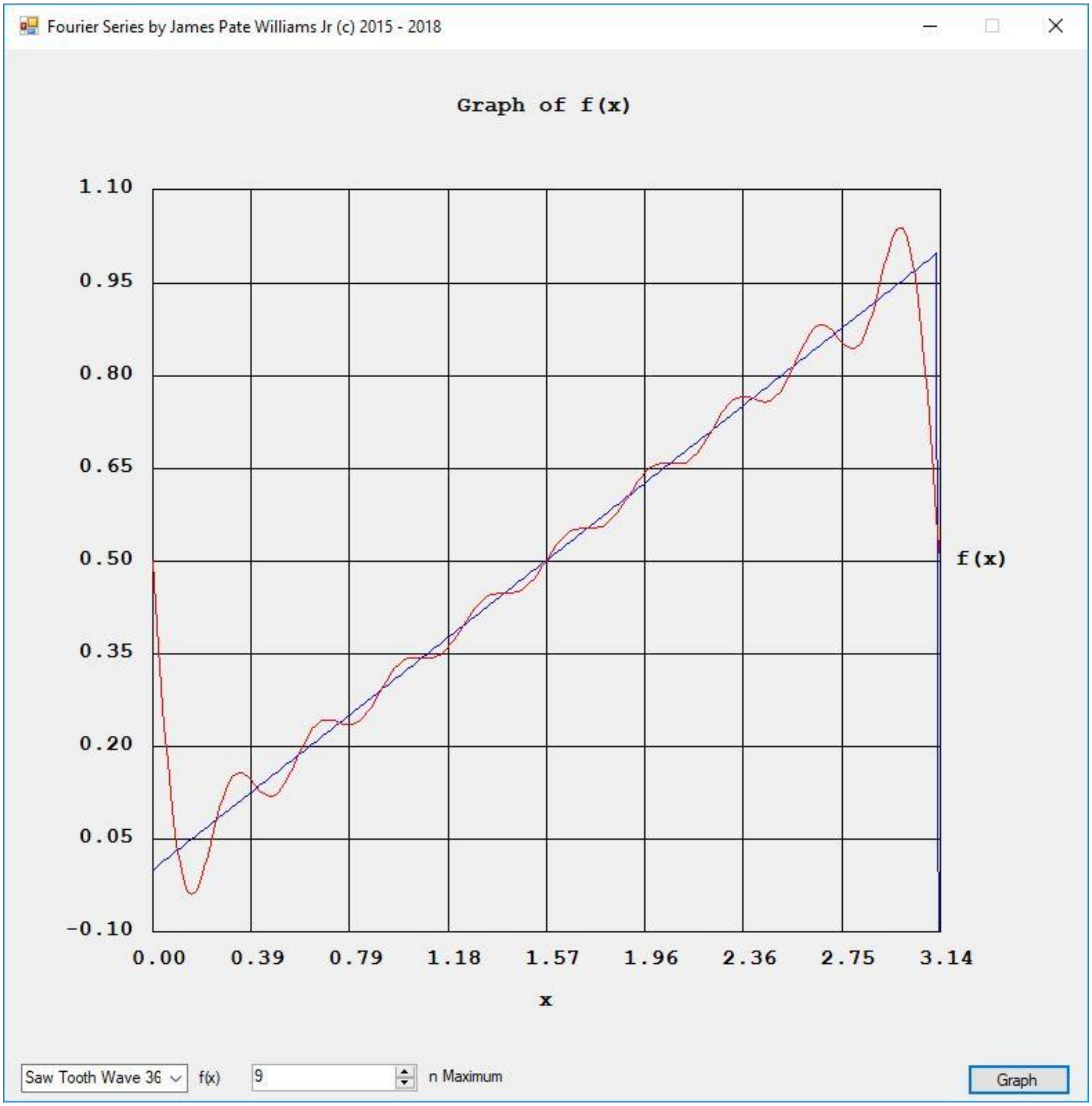

Infinite Series by James Pate Williams Jr (c) 2018
Exercise 36.3 (b) (1)
n Maximum = 100
Approximate Value = 1.0146780114
True Value = 1.0146780316
% Error = 0.00000199
Exercise 36.3 (b) (1)
n Maximum = 500
Approximate Value = 1.0146780314
True Value = 1.0146780316
% Error = 0.00000002
Exercise 36.3 (b) (1)
n Maximum = 1000
Approximate Value = 1.0146780316
True Value = 1.0146780316
% Error = 0.00000000
Exercise 36.3
Exercise #
1000
n Maximum
Clear
Solve


Infinite Series by James Pate Williams Jr (c) 2018
Exercise 36.3 (b) (1)
n Maximum = 100
Approximate Value = 1.0823229053
True Value = 1.0823232337
% Error = 0.00003034
Exercise 36.3 (b) (1)
n Maximum = 500
Approximate Value = 1.0823232311
True Value = 1.0823232337
% Error = 0.00000025
Exercise 36.3 (b) (1)
n Maximum = 1000
Approximate Value = 1.0823232334
True Value = 1.0823232337
% Error = 0.00000003
Exercise 36.3 (b) (1)
n Maximum = 1500
Approximate Value = 1.0823232336
True Value = 1.0823232337
% Error = 0.00000001
Exercise 36.3 (b) (1)
n Maximum = 2000
Approximate Value = 1.0823232337
True Value = 1.0823232337
% Error = 0.00000000
Exercise 36.3
Exercise #
2000
n Maximum
Clear
Solve

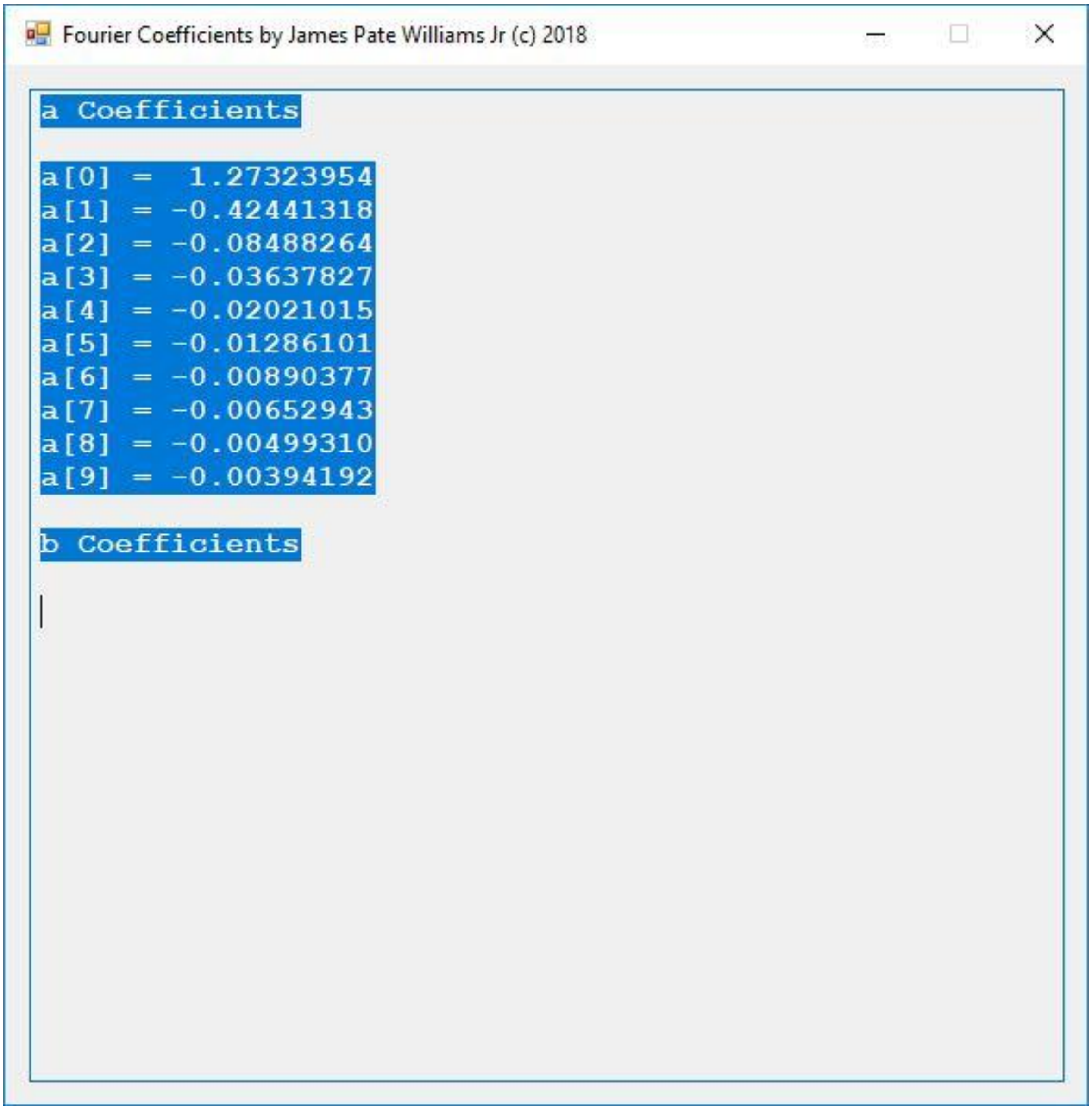

Infinite Series by James Pate Williams Jr (c) 2018
Exercise 36.4
n Maximum = 1000
Integral Value = 5.999999998014
Series Sum Value = 1.082323233378
Approximate Value = 6.4939393981
True Value = 6.4939394023
% Error = 0.00000006
Exercise 36.4
n Maximum = 1500
Integral Value = 5.999999998014
Series Sum Value = 1.082323233612
Approximate Value = 6.4939393995
True Value = 6.4939394023
% Error = 0.00000004
Exercise 36.4
n Maximum = 2000
Integral Value = 5.999999998014
Series Sum Value = 1.082323233670
Approximate Value = 6.4939393999
True Value = 6.4939394023
% Error = 0.00000004
Exercise 36.4
n Maximum = 2500
Integral Value = 5.999999998014
Series Sum Value = 1.082323233690
Approximate Value = 6.4939394000
True Value = 6.4939394023
% Error = 0.00000004
Exercise 36.4
Exercise #
2500
n Maximum
Clear
Solve

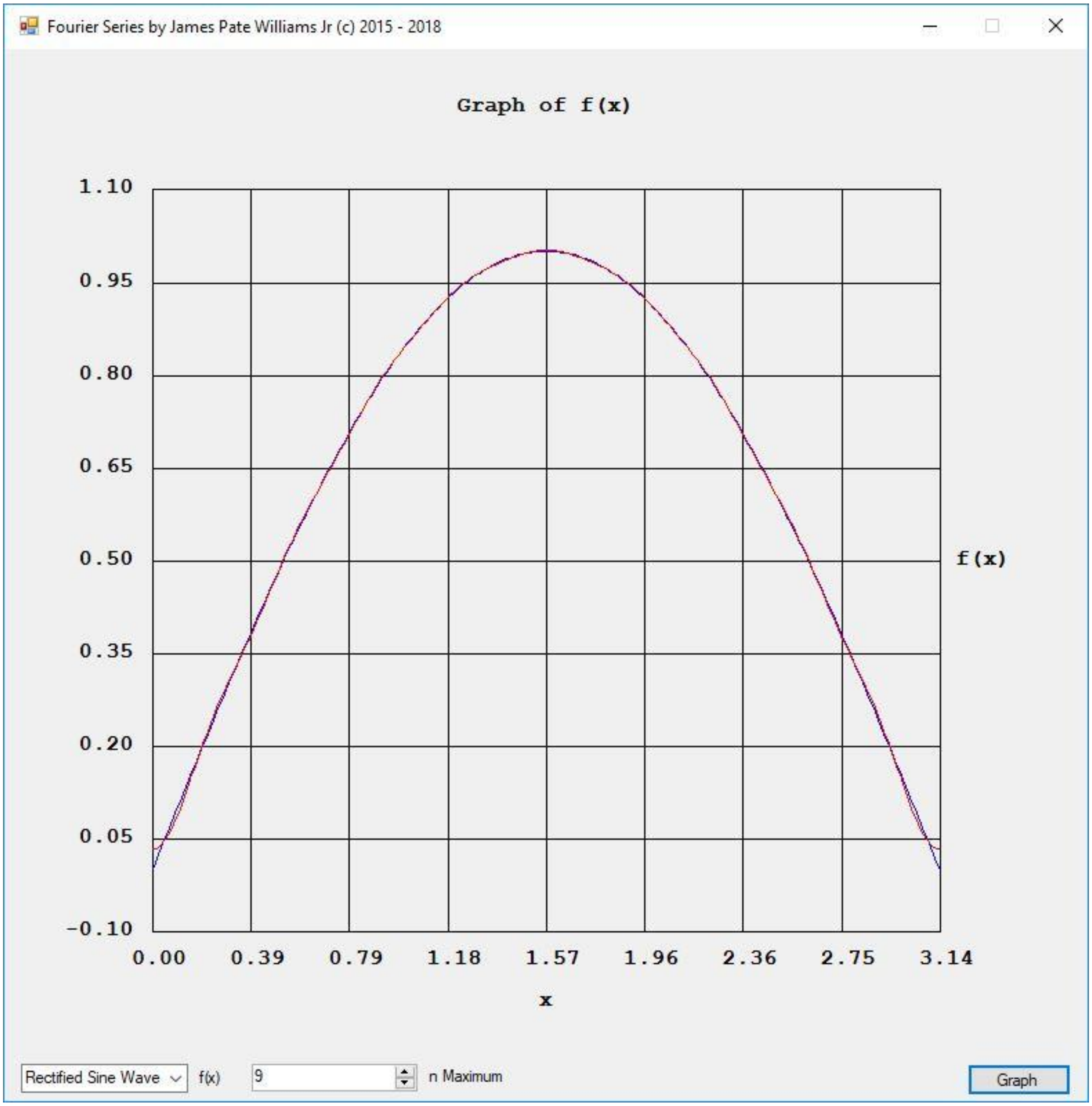
Fourier coefficients and graph for Exercise 36.6, the sawtooth wave.




Fourier Series by James Pate Williams Jr (c) 2015 - 2018
Graph of f(x)
1.10
0.95
0.80
0.65
0.50
0.35
0.20
0.05
-0.10
0.00
0.39
0.79
1.18
1.57
1.96
2.36
2.75
3.14
x
f(x)
Saw Tooth Wave 36
f(x)
9
n Maximum
Graph

Fourier coefficients and graph for Exercise 36.8, the rectified sine wave.




Fourier Series by James Pate Williams Jr (c) 2015 - 2018
Graph of f(x)
1.10
0.95
0.80
0.65
0.50
0.35
0.20
0.05
-0.10
0.00
0.39
0.79
1.18
1.57
1.96
2.36
2.75
3.14
x
f(x)
Rectified Sine Wave
f(x)
9
n Maximum
Graph