Exercises for the Feynman Lectures on Physics by Richard Feynman, Et Al. Chapter 38 Differential Calculus of Vector Fields – Detailed Work by James Pate Williams, Jr. BA, BS, MSwE, PhD

38.2

(a)

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = (\partial_x, \partial_y, \partial_z)\cdot\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \partial_x & \partial_y & \partial_z\\ A_x & A_y & A_z\end{vmatrix}$$

$$= \partial_x(\partial_y A_z - \partial_z A_y) + \partial_y(\partial_z A_x - \partial_x A_z) + \partial_z(\partial_x A_y - \partial_y A_x)$$

$$= \partial_x\partial_y A_z - \partial_y\partial_x A_z + \partial_y\partial_z A_x - \partial_z\partial_y A_x + \partial_z\partial_x A_y - \partial_x\partial_z A_y = 0$$

$$[\partial_x, \partial_y] = \partial_x\partial_y - \partial_y\partial_x = 0$$

$$\partial_x = \frac{\partial}{\partial x}$$

(b)

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \partial_x & \partial_y & \partial_z\\ \partial_y A_z - \partial_z A_y & \partial_z A_x - \partial_x A_z & \partial_x A_y - \partial_y A_x\end{vmatrix}$$

$$= \partial_x(\partial_y A_z - \partial_z A_y)\hat{\imath} + \partial_y(\partial_z A_x - \partial_x A_z)\hat{\jmath} + \partial_z(\partial_x A_y - \partial_y A_x)\hat{k}$$

$$\vec{\nabla}\times\vec{A} = \varepsilon_{ijk}\partial_i A_j \hat{e}_k$$

$$\vec{\nabla}\cdot\vec{A} = \partial_i A_i$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \varepsilon_{ijk}\partial_i(\vec{\nabla}\times\vec{A})_j\hat{e}_k = \varepsilon_{ijk}\partial_i\varepsilon_{lmj}\partial_l A_m\hat{e}_k = \varepsilon_{ijk}\varepsilon_{lmj}\partial_i\partial_l A_m\hat{e}_k$$

$$\varepsilon_{ijk} = \begin{cases} -1\forall ijk\ is\ an\ odd\ permutation\\ 0\forall i = j\ or\ i = k\ or\ j = k\\ +1\forall ijk\ is\ an\ even\ permutation\end{cases}$$

The following identity is from the online paper [1]:

$$\varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$\varepsilon_{ijk}\varepsilon_{lmj} = -\varepsilon_{ikj}\varepsilon_{lmj} = -\delta_{il}\delta_{km} + \delta_{im}\delta_{kl}$$

$$\begin{aligned}\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) &= (\delta_{il}\delta_{km} - \delta_{im}\delta_{kl})\partial_i\partial_l A_m\hat{e}_k = -\delta_{il}\delta_{km}\partial_i\partial_l A_m\hat{e}_k + \delta_{im}\delta_{kl}\partial_i\partial_l A_m\hat{e}_k \\ &= -\partial_i\partial_i A_k\hat{e}_k + \partial_i\partial_l A_i\hat{e}_l = -(\vec{\nabla}\cdot\vec{\nabla})\vec{A} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right) = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right) - \nabla^2\vec{A}\end{aligned}$$

38.2 See [2].

(a)

$$\vec{R} = (x \quad y \quad z) = (x_1 \quad x_2 \quad x_3)$$

$$\vec{\nabla}\cdot\vec{R} = \partial_{x_i}x_i = 1+1+1 = 3$$

(b)

$$\vec{\nabla}\times\vec{R} = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \partial_x & \partial_y & \partial_z \\ x & y & z\end{vmatrix} = \partial_x\left(\partial_y z - \partial_z y\right)\hat{\imath} + \partial_y(\partial_z x - \partial_x z)\hat{\jmath} + \partial_z\left(\partial_x y - \partial_y x\right)\hat{k} = \vec{0}$$

Show that for all R not equal zero:

(c)

$$\vec{\nabla}\cdot\left(\frac{\vec{R}}{R^3}\right) = \partial_{x_i}\left(\frac{x_i}{R^3}\right) = \frac{\vec{\nabla}\cdot\vec{R}}{R^3} + x_i\partial_{x_i}\left(\frac{1}{R^3}\right) = \frac{3}{R^3} - \frac{3x_ix_i}{R^5} = \frac{3}{R^3} - \frac{3R^2}{R^5} = \frac{3}{R^3} - \frac{3}{R^3} = 0$$

(d)

$$\vec{\nabla}\times\left(\frac{\vec{R}}{R^3}\right) = \varepsilon_{ijk}\partial_i\left(\frac{x_j}{R^3}\right)\hat{e}_k = \frac{1}{R^3}\varepsilon_{ijk}\delta_{ij}\hat{e}_k - 3\varepsilon_{ijk}\frac{x_ix_j}{R^5}\hat{e}_k = \left(\frac{3}{R^3} - \frac{3x_ix_i}{R^5}\right)\hat{e}_k = \vec{0}$$

(e)

$$\vec{\nabla}\left(\frac{1}{R}\right) = \partial_{x_i}\left(\frac{1}{R}\right)\hat{e}_i = -\frac{x_i}{R^3}\hat{e}_i = -\frac{\vec{R}}{R^3}$$

(f)

$$\vec{R} = \vec{\nabla}\varphi \rightarrow \varphi = \frac{1}{2}\left(\vec{R}\cdot\vec{R}\right) = \frac{1}{2}(x_ix_i)$$

$$\vec{\nabla}\varphi = \frac{1}{2}\partial_{x_i}(x_kx_k)\hat{e}_i = \frac{1}{2}(2\delta_{ik}x_k)\hat{e}_i = x_i\hat{e}_i = \vec{R}$$

38.3

(a)

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{E}\right) = -\frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{B}\right)$$

From Exercise **38.2** (a) we find:

$$0 = -\frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{B}\right) \therefore \vec{\nabla}\cdot\vec{B} = 0$$

(b)

$$\vec{\nabla}\cdot\left[c^2\left(\vec{\nabla}\times\vec{B}\right)\right] = \vec{\nabla}\cdot\left(\frac{\partial\vec{E}}{\partial t}+\frac{\vec{j}}{\epsilon_0}\right) = c^2\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{B}\right) = 0 = \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{E}\right)+\frac{\vec{\nabla}\cdot\vec{j}}{\epsilon_0}$$

$$\frac{1}{\epsilon_0}\frac{\partial \mathrm{\rho}}{\partial \mathrm{t}}+\frac{\vec{\nabla}\cdot\vec{j}}{\epsilon_0} = 0$$

$$\vec{\nabla}\cdot\vec{j} = -\frac{\partial \mathrm{\rho}}{\partial \mathrm{t}}$$

(c)

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) = -\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right)$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right)-\nabla^2\vec{E} = -\nabla^2\vec{E}$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial \mathrm{t}}$$

$$-\nabla^2\vec{E} = -\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\nabla^2\vec{E}-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

(d)

$$\vec{\nabla}\times\left[c^2\left(\vec{\nabla}\times\vec{B}\right)\right] = \frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{E}\right)$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right) = \frac{1}{c^2}\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{E}\right) = -\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right) = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{B}\right)-\nabla^2\vec{B} = -\nabla^2\vec{B}$$

$$\nabla^2\vec{B}-\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = 0$$

(e)

$$\vec{E} = -\vec{\nabla}\phi-\frac{\partial\vec{A}}{\partial \mathrm{t}}$$

$$\vec{\nabla}\times\vec{E} = -\vec{\nabla}\times\vec{\nabla}\phi-\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{A}\right) = -\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{A}\right) = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

(f)

Since the following relationship holds:

$$\vec{\nabla} \cdot \vec{B} = 0$$

By a previous result then by second exercise part (a) we have:

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

38.5

(a)

$$\overrightarrow{v(x, y, z)} = \vec{\omega} \times \vec{r}$$

(b)

$$\vec{\nabla} \cdot \overrightarrow{v(x, y, z)} = \vec{\nabla} \cdot (\vec{\omega} \times \vec{r}) = 0$$

$$\vec{\nabla} \times (\vec{\omega} \times \vec{r}) = \vec{\omega}(\vec{\nabla} \cdot \vec{r}) - \vec{r}(\vec{\nabla} \cdot \vec{\omega}) + (\vec{r} \cdot \vec{\nabla})\vec{\omega} - (\vec{\omega} \cdot \vec{\nabla})\vec{r} = 3\vec{\omega} - \vec{\omega} = 2\vec{\omega}$$

The angular velocity is a constant vector.

38.6

(a)

$$\vec{A} \times \vec{R} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ x & y & z \end{vmatrix} = (A_y z - A_z y)\hat{\imath} + (A_z x - A_x z)\hat{\jmath} + (A_x y - A_y x)\hat{k}$$

$$\vec{\nabla} \times (\vec{A} \times \vec{R}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ A_y z - A_z y & A_z x - A_x z & A_x y - A_y x \end{vmatrix} = (A_x + A_x)\hat{\imath} + (A_y + A_y)\hat{\jmath} + (A_z + A_z)\hat{k}$$
$$= 2\vec{A}$$

(b)

$$\vec{B} \times (\vec{A} \times \vec{C}) = \vec{A}(\vec{B} \cdot \vec{C}) - \vec{C}(\vec{\mathrm{B}} \cdot \vec{A})$$

$$\vec{\nabla} \times (\vec{A} \times \vec{C}) = \vec{A}(\vec{\nabla} \cdot \vec{C}) - \vec{C}(\vec{\nabla} \cdot \vec{A})$$

$$\vec{\nabla} \times (\vec{A} \times \vec{R}) = \vec{A}(\vec{\nabla} \cdot \vec{R}) - \vec{R}(\vec{\nabla} \cdot \vec{A}) = 3\vec{A}$$

The preceding equation is only half the complete equation:

$$\vec{\nabla} \times (\vec{A} \times \vec{R}) = \vec{A}(\vec{\nabla} \cdot \vec{R}) - \vec{R}(\vec{\nabla} \cdot \vec{A}) + (\vec{R} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{R} = 3\vec{A} - \vec{A} = 2\vec{A}$$

References

[1] P. Renteln, "Lecture on Vector Calculus," Department of Physics California State University, March 2011. [Online]. Available: http://physics.csusb.edu/~prenteln/notes/vc_notes.pdf. [Accessed 8 May 2018].

[2] J. D. Jackson, "Vector Formulas," in *Classical Electrodynmaics Second Edition*, New York, John Wiley & Sons, 1975, p. Front Inside Book Cover.