Stellar Atmospheres (Second Edition)
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Exercise 1-3: Derive Stefan's law by substituting $x \equiv h\nu/kT$, and expanding $(e^x-1)^{-1} = e^{-x}(1-e^{-x})^{-1}$ as a power series in e^{-x}. The sum obtained from the term-by-term integration is related to the Riemann Zeta-function [See (4.807)].

$$E_R^* = \frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3 d\nu}{e^{h\nu/kT}-1)} = a_R T^4 \text{ (Stefan's Law)}$$

$$a_R = \frac{8\pi^5 k^4}{15c^3h^3}$$

$$f(\nu) = \frac{1}{e^{h\nu/kT}-1}$$

$$g(x) = \frac{1 \cdot e^{-x}}{(e^x-1)\cdot e^{-x}} = \frac{e^{-x}}{(1-e^{-x})} \qquad x = \frac{h\nu}{kT}$$

$$H(Y) = \frac{Y}{(1-Y)} = Y(1-Y)^{-1} \qquad Y = e^{-x}$$

$$H'(Y) = (1-Y)^{-1} + Y(1-Y)^{-2}$$

$$H''(Y) = (1-Y)^{-2} + (1-Y)^{-2} + 2Y(1-Y)^{-3}$$
$$= 2(1-Y)^{-2} + 2Y(1-Y)^{-3}$$

$$H'''(Y) = 4(1-Y)^{-3} + 2(1-Y)^{-3} + 6Y(1-Y)^{-4}$$
$$= 6(1-Y)^{-3} + 6Y(1-Y)^{-4}$$

$$H^{iv}(Y) = 18(1-Y)^{-4} + 6(1-Y)^{-4} + 24Y(1-Y)^{-5}$$
$$= 24(1-Y)^{-4} + 24Y(1-Y)^{-5}$$

$$H^{v}(Y) = 96(1-Y)^{-5} + 24(1-Y)^{-5} + 120Y(1-Y)^{-6}$$
$$= 120(1-Y)^{-5} + 120Y(1-Y)^{-6}$$

$$H^n(y) = n!(1-y)^{-n} + n!\,y(1-y)^{-n-1}$$

$$H(0) = 0$$

$$H'(0) = 1$$

$$H''(0) = 2$$

$$H'''(0) = 6$$

$$H^{iv}(0) = 24$$

$$H^{v}(0) = 120$$

$$H^n(0) = n!$$

$$H(y) = \sum_{n=1}^{\infty} \frac{H^n(0)}{n!} y^n = \sum_{n=1}^{\infty} \frac{n!}{n!} y^n = \sum_{n=1}^{\infty} y^n$$

$$g(x) = \sum_{n=1}^{\infty} e^{-nx}$$

$$f(\nu) = \sum_{n=1}^{\infty} e^{-nh\nu/kT}$$

$$E_R^* = \frac{8\pi h}{c^3} \sum_{n=1}^{\infty} \int_0^{\infty} \nu^3 e^{-nh\nu/kT}\, d\nu$$ termwise integrate by parts

$$\int_0^{\infty} \nu^3 e^{-nh\nu/kT}\, d\nu = -\frac{kT}{nh}\nu^3 e^{-nh\nu/kT}\Big|_{\nu=0}^{\nu\to\infty}$$

$$- \frac{3k^2T^2}{n^2h^2}\nu^2 e^{-nh\nu/kT}\Big|_{\nu=0}^{\nu\to\infty}$$

$$- \frac{6k^3T^3}{n^3h^3}\nu e^{-nh\nu/kT}\Big|_{\nu=0}^{\nu\to\infty}$$

$$- \frac{6k^4T^4}{n^4h^4} e^{-nh\nu/kT}\Big|_{\nu=0}^{\nu\to\infty}$$

$$= \frac{6k^4T^4}{n^4h^4}$$

$$E_R^* = \frac{8\pi h}{c^3} \cdot \frac{6k^4T^4}{h^4} \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{48\pi k^4T^4}{c^3h^3}\zeta(4) = \frac{48\pi k^4T^4}{c^3h^3}\left(\frac{\pi^4}{90}\right)$$

see Wikipedia Riemann zeta-function article

$$E_R^* = \frac{6 \cdot 8\pi^5 k^4}{6 \cdot 15 c^3 h^3} T^4 = \frac{8\pi^5 k^4}{15 c^3 h^3} T^4 = a_R T^4 \checkmark \quad (\text{Q.E.D.})$$

Problem The relativistic energy of a particle is given by (Einstein 1905)

$$E = mc^2 = m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

Find the classical limit of E.

Let $\beta = \frac{v}{c}$

$$E = m_0 c^2 (1 - \beta^2)^{-1/2}$$

$$f(\beta) = (1 - \beta^2)^{-1/2}$$

$$f'(\beta) = \beta(1 - \beta^2)^{-3/2}$$

$$f''(\beta) = (1 - \beta^2)^{-3/2} + 3\beta^2(1 - \beta^2)^{-5/2}$$

$$f'''(\beta) = 3(1 - \beta^2)^{-5/2}\beta + 6\beta(1 - \beta^2)^{-5/2} + 15\beta^3(1 - \beta^2)^{-7/2}$$

$$= 9\beta(1 - \beta^2)^{-5/2} + 15\beta^3(1 - \beta^2)^{-7/2}$$

$$f^{iv}(\beta) = 9(1 - \beta^2)^{-5/2} + 45\beta^2(1 - \beta^2)^{-7/2} + 45\beta^2(1 - \beta^2)^{-7/2}$$

$$+ 105\beta^4(1 - \beta)^{-9/2}$$

$$f^{v}(\beta) = 45\beta(1 - \beta^2)^{-7/2} + 90\beta(1 - \beta^2)^{-7/2} + 90\beta(1 - \beta^2)^{-7/2}$$

$$+ 315\beta^3(1 - \beta^2)^{-9/2} + 420\beta^3(1 - \beta^2)^{-9/2} + 945\beta^5(1 - \beta^2)^{-11/2}$$

$f(0) = 1$

$f'(0) = 0$

$f''(0) = 1$

$f'''(0) = 0$

$f^{IV}(0) = 9$

$$f(\beta) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}\beta^n = 1 + \frac{1}{2}\beta^2 + \frac{9}{6}\beta^4 + \cdots$$

$$E = mc^2 = m_0c^2 f(\beta) = m_0c^2 + m_0c^2 \frac{1}{2}\frac{v^2}{c^2}$$

$$+ \frac{3}{2} m_0c^2 \frac{v^4}{c^4} + \cdots$$

$$= m_0c^2 + \frac{1}{2}m_0v^2 + \frac{3}{2}m_0\frac{v^4}{c^2} + \cdots$$

Suppose $v \ll c$

$E \simeq m_0c^2 + \frac{1}{2}m_0v^2$ (Classical limit)