The time dependent Schrödinger equation for the hydrogen-like atom is [1]

$$\frac{ih}{2\pi}\frac{\partial}{\partial t}\Psi(\vec{r_n},\vec{r_e},t) = H\Psi(\vec{r_n},\vec{r_e},t)$$

Where the classical Hamiltonian for a single electron atom is given by:

$$H = \frac{p_n^2}{2m_n} + \frac{p_e^2}{2m_e} - \frac{Ze^2}{\|\vec{r_n} - \vec{r_e}\|^{1/2}}$$

$$p_n^2 = p_{nX}^2 + p_{nY}^2 + p_{nZ}^2$$

$$p_e^2 = p_{eX}^2 + p_{eY}^2 + p_{eZ}^2$$

$$\|\vec{r_n} - \vec{r_e}\|^{1/2} = \sqrt{(x_n - x_e)^2 + (y_n - y_e)^2 + (z_n - z_e)^2}$$

We transform from classical momenta to quantum mechanics as follows:

$$p_n^2 = -\frac{h^2}{4\pi^2}\nabla_n{}^2 = -\frac{h^2}{4\pi^2}\left(\frac{\partial^2}{\partial x_n{}^2} + \frac{\partial^2}{\partial y_n{}^2} + \frac{\partial^2}{\partial z_n{}^2}\right)$$

Next, we transform from laboratory frame of reference to the center of mass frame of reference.

$$x = x_n - x_e$$

$$y = y_n - y_e$$

$$z = z_n - z_e$$

$$r = \|\vec{r_n} - \vec{r_e}\|^{1/2} = \sqrt{x^2 + y^2 + z^2}$$

$$X_n = x_n - X$$

$$X_e = x_e - X$$

$$m_n X_n + m_e X_e = m_n x_n + m_e x_e - m_n X - m_e X = 0$$

$$(m_n + m_e)X = MX = m_n x_n + m_e x_e$$

$$(m_n + m_e)Y = MY = m_n y_n + m_e y_e$$

$$(m_n + m_e)Z = MZ = m_n z_n + m_e z_e$$

Eliminate the electron position from the x and X equations by addition of the following two equations:

$$m_e x = m_e x_n - m_e x_e$$

$$MX = m_n x_n + m_e x_e$$

$$m_e x + MX = m_e x_n + m_n x_n = (m_e + m_n)x_n = Mx_n$$

$$x_n = \frac{m_e x}{M} + X$$

$$x_e = x_n + x = \frac{m_e x}{M} + X + x$$

$$dx_n = \frac{m_e dx}{M} + dX$$

$$dx_e = \frac{m_e dx}{M} + dX + dx$$

$$\frac{\partial x_n}{\partial x} = \frac{m_e}{M}$$

$$\frac{\partial x_n}{\partial X} = 1$$

$$\frac{\partial x_e}{\partial x} = \frac{m_e}{M} + 1 = \frac{m_n + 2m_e}{M}$$

$$\frac{\partial x_e}{\partial X} = 1$$

$$\frac{\partial}{\partial x_n} = \frac{\partial x}{\partial x_n}\frac{\partial}{\partial x} + \frac{\partial X}{\partial x_n}\frac{\partial}{\partial X}$$

$$\frac{\partial^2}{\partial x_n^2} = \frac{\partial x}{\partial x_n}\frac{\partial^2}{\partial x^2} + \frac{\partial X}{\partial x_n}\frac{\partial^2}{\partial X^2} = \frac{M}{m_e}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial X^2}$$

$$\frac{\partial^2}{\partial x_e^2} = \frac{\partial x}{\partial x_e}\frac{\partial^2}{\partial x^2} + \frac{\partial X}{\partial x_e}\frac{\partial^2}{\partial X^2} = \frac{M}{m_n + 2m_e}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial X^2}$$

$$p_n^2 = -\frac{h^2}{4\pi^2}{\nabla_n}^2 = -\frac{h^2}{4\pi^2}\frac{M}{m_e}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) - \frac{h^2}{4\pi^2}\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2}\right)$$

$$p_e^2 = -\frac{h^2}{4\pi^2}{\nabla_e}^2 = -\frac{h^2}{4\pi^2}\frac{M}{m_n + 2m_e}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) - \frac{h^2}{4\pi^2}\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2}\right)$$

$$H = -\frac{h^2}{8\pi^2}\frac{M}{m_n m_e}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) - \frac{h^2}{8\pi^2 m_n}\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2}\right) - \frac{h^2}{8\pi^2}\frac{M}{m_e(m_n + 2m_e)}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) - \frac{h^2}{8\pi^2 m_e}\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2}\right)$$

$$H = -\frac{h^2}{8\pi^2\mu}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) - \frac{h^2}{8\pi^2 M}\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2}\right)$$

$$\Psi(\overrightarrow{r_n}, \overrightarrow{r_e}, t) = \Psi(\vec{r}, \vec{R}, t) = \psi(\vec{r})U(\vec{R})T(t)$$

$$\frac{ih}{2\pi T}\frac{dT}{dt} = -\frac{h^2}{8\pi^2\mu\psi}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) - \frac{h^2}{8\pi^2 MU}\left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} + \frac{\partial^2 U}{\partial Z^2}\right) - \frac{Ze^2}{r} = E + E'$$

$$\frac{dT}{T} = -\frac{2\pi i}{h}(E + E')dt$$

$$-\frac{h^2}{8\pi^2\mu}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) - \frac{Ze^2}{r}\psi = E\psi$$

$$-\frac{h^2}{8\pi^2 M}\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}+\frac{\partial^2 U}{\partial Z^2}\right)=E'U$$

$$T(t)=e^{-2\pi i(E+E\prime)t/h}$$

$$U(X,Y,Z)=u(X)v(Y)w(Z)$$

$$\frac{1}{u}\frac{d^2u}{dX^2}+\frac{1}{v}\frac{d^2v}{dY^2}+\frac{1}{w}\frac{d^2w}{dZ^2}=-\frac{8\pi^2 ME'}{h^2}$$

$$E'=E_X+E_Y+E_Z$$

$$\frac{1}{u}\frac{d^2u}{dX^2}=-\frac{8\pi^2 ME_X}{h^2}$$

$$\frac{1}{v}\frac{d^2v}{dY^2}=-\frac{8\pi^2 ME_Y}{h^2}$$

$$\frac{1}{w}\frac{d^2w}{dZ^2}=-\frac{8\pi^2 ME_Z}{h^2}$$

$$\frac{d^2u}{dX^2}+\frac{8\pi^2 ME_X}{h^2}u=0$$

$$\frac{d^2v}{dY^2}+\frac{8\pi^2 ME_Y}{h^2}v=0$$

$$\frac{d^2w}{dZ^2}+\frac{8\pi^2 ME_Z}{h^2}w=0$$

$$\frac{d^2u}{dX^2}+\alpha^2u=0$$

$$\frac{d^2v}{dY^2}+\beta^2v=0$$

$$\frac{d^2w}{dZ^2}+\gamma^2w=0$$

$$\alpha^2=\frac{8\pi^2 ME_X}{h^2}$$

$$\beta^2=\frac{8\pi^2 ME_Y}{h^2}$$

$$\gamma^2=\frac{8\pi^2 ME_Z}{h^2}$$

$$u(X)=A_X\sin(\alpha X)+B_X\cos(\alpha X)$$

$$v(Y)=A_Y\sin(\beta Y)+B_Y\cos(\beta Y)$$

$$w(Z)=A_Z\sin(\gamma Z)+B_Z\cos(\gamma Z)$$

References

[1] L. I. Schiff, "Chapter 4 Discrete Eigenvalues: Bound States Section 16 The Hydrogen Atom," in *Quantum Mechanics Third Edition*, New York, McGraw-Hill Inc., 1968, pp. 88-90.