The Laplace Equation in Various Orthogonal Curvilinear Coordinate Systems

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Solve the Laplace equation in the following orthogonal curvilinear coordinate systems:

a. Cartesian coordinates
b. Cylindrical coordinates
c. Spherical polar coordinates
d. Parabolic cylindrical coordinates

The Laplacian operator in orthogonal curvilinear coordinates is defined by the equation:

$$\Delta = \vec{\nabla} \cdot \vec{\nabla} = \nabla^2 = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial}{\partial u_3}\right)\right]$$

The line element is defined in terms of the metric tensor and scale factors as:

$$ds^2 = g_{mn} du^m du^n$$

$$h_m = \sqrt{g_{mm}}$$

a. Cartesian coordinates

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The potential equation is given by:

$$\nabla^2 \Phi(x, y, z) = 0$$

$$\Phi(0,0,0) = \Phi(a, b, c) = 0$$

$$\Phi(x, y, z) = X(x)Y(y)Z(z)$$

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = -k^2 - l^2 - m^2$$

$$X''(x) + k^2 X(x) = 0$$

$$Y''(y) + l^2 Y(y) = 0$$

$$Z''(z) + m^2 Z(z) = 0$$

$$X(x) = A\sin(kx)$$

$$\sin(ka) = 0 \to ka = n_x\pi, k = \frac{n_x\pi}{a} \forall n_x \in \{1,2,\cdots\}$$

$$\int_0^a X^2(x)dx = A^2 \int_0^a \sin\left(\frac{n_x\pi x}{a}\right)^2 dx$$

$$\left(e^{ix}\right)^2 = e^{2ix} = \cos(2x) + i\sin(2x) = (\cos x + i\sin x)^2 = (\cos x)^2 + i^2(\sin x)^2 + 2i\sin x\cos x$$

$$\cos(2x) = (\cos x)^2 - (\sin x)^2 = 1 - 2(\sin x)^2$$

$$(\sin x)^2 = \frac{1}{2}[1 - \cos(2x)]$$

$$\int_0^a \sin\left(\frac{n_x\pi x}{a}\right)^2 dx = \frac{1}{2}\int_0^a dx - \frac{1}{2}\int_0^a \cos\left(\frac{2n_x\pi x}{a}\right) dx = \frac{a}{2} - \frac{1}{2}\cdot\frac{a}{2n_x\pi}\int_0^a d\sin\left(\frac{2n_x\pi x}{a}\right) = \frac{a}{2}$$

$$A = \sqrt{\frac{2}{a}}$$

$$\Phi(x,y,z) = \sqrt{\frac{8}{abc}}\sum_{n_x=1}^{\infty}\sum_{n_y=1}^{\infty}\sum_{n_z=1}^{\infty} \sin\left(\frac{n_x\pi x}{a}\right)\sin\left(\frac{n_y\pi y}{b}\right)\sin\left(\frac{n_z\pi z}{c}\right)$$

b. Cylindrical coordinates

$$x = \rho\cos\phi$$

$$y = \rho\sin\phi$$

$$z = z$$

$$ds^2 = dx^2 + dy^2 + dz^2 = (\cos\phi\, d\rho - \rho\sin\phi\, d\phi)^2 + (\sin\phi d\rho + \rho\cos\phi\, d\phi)^2 + dz^2 = (\cos\phi)^2 d\rho^2 + (\sin\phi)^2 d\rho^2 - 2\rho\sin\phi\cos\phi\, d\rho d\phi + 2\rho\sin\phi\cos\phi\, d\rho d\phi + \rho^2[(\cos\phi)^2 + (\sin\phi)^2]d\phi^2 + dz^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$$

$$h_\rho = h_z = 1, h_\phi = \rho$$

$$\nabla^2 = \frac{1}{h_\rho h_\phi h_z}\left[\frac{\partial}{\partial\rho}\left(\frac{h_\phi h_z}{h_\rho}\frac{\partial}{\partial\rho}\right) + \frac{\partial}{\partial\phi}\left(\frac{h_\rho h_z}{h_\phi}\frac{\partial}{\partial\phi}\right) + \frac{\partial}{\partial z}\left(\frac{h_\rho h_\phi}{h_z}\frac{\partial}{\partial z}\right)\right] = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho}\frac{\partial^2}{\partial\phi^2} + \rho\frac{\partial^2}{\partial z^2}\right]$$

$$\nabla^2\Phi(\rho,\phi,z) = 0$$

$$\Phi(0,0,0) = \Phi(r,2\pi,h) = 0$$

$$\Phi(\rho,\phi,z) = R(\rho)Q(\phi)Z(z)$$

$$\frac{1}{\rho}\cdot\frac{1}{R}\cdot\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\frac{1}{\rho^2}\cdot\frac{1}{Q}\cdot\frac{dQ^2}{d\phi^2}+\frac{1}{Z}\cdot\frac{d^2Z}{dz^2} = -k^2-l^2$$

$$\frac{1}{\rho}\cdot\frac{1}{R}\cdot\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\frac{1}{\rho^2}\cdot\frac{1}{Q}\cdot\frac{dQ^2}{d\phi^2} = -k^2$$

$$\frac{1}{Z}\cdot\frac{d^2Z}{dz^2} = -l^2$$

$$\frac{\rho}{R}\cdot\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\frac{1}{Q}\cdot\frac{dQ^2}{d\phi^2} = -k^2\rho^2$$

$$\frac{\rho}{R}\cdot\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+k^2\rho^2+\frac{1}{Q}\cdot\frac{dQ^2}{d\phi^2} = m^2$$

$$\frac{\rho}{R}\cdot\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+k^2\rho^2-m^2 = 0$$

$$\frac{1}{Q}\cdot\frac{dQ^2}{d\phi^2} = m^2$$

$$\rho^2R''(\rho)+\rho R'(\rho)+k^2\rho^2R(\rho)-m^2R(\rho) = 0, R(0) = R(r) = 0$$

$$Q''(\phi)-m^2Q(\phi) = 0, Q(0) = Q(2\pi) = 0$$

$$Z''(z)+l^2Z(z) = 0, Z(0) = Z(h) = 0$$

$$R(\rho) = \sum_{j=0}^{\infty}a_j\rho^{\alpha+j}$$

$$R'(\rho) = \sum_{j=0}^{\infty}(\alpha+j)a_j\rho^{\alpha+j-1}$$

$$R''(\rho) = \sum_{j=0}^{\infty}(\alpha+j)(\alpha+j-1)a_j\rho^{\alpha+j-2}$$

$$\sum_{j=0}^{\infty}(\alpha+j)(\alpha+j-1)a_j\rho^{\alpha+j}+\sum_{j=0}^{\infty}(\alpha+j)a_j\rho^{\alpha+j}+\sum_{j=0}^{\infty}k^2a_j\rho^{\alpha+j+2}-\sum_{j=0}^{\infty}m^2a_j\rho^{\alpha+j}=0$$

$$a_0[\alpha(\alpha-1)+\alpha-m^2]=a_0(\alpha^2-m^2)=0\to\alpha=m$$

$$a_1[(\alpha+1)(\alpha+1-1)+\alpha+1-m^2]=a_1(\alpha^2+2\alpha+1-m^2)=a_1[(\alpha+1)^2-m^2]=0$$

$$\begin{aligned}&a_2[(\alpha+2)(\alpha+2-1)+(\alpha+2)-m^2]+a_0k^2\\&=a_2[\alpha^2+2\alpha-\alpha+2\alpha+4-2+\alpha+2-m^2]+a_0k^2\\&=a_2[\alpha^2+4\alpha+4-m^2]+a_0k^2=a_2[(\alpha+2)^2-m^2]+a_0k^2=0\end{aligned}$$

$$a_j[(\alpha+j)^2-m^2]+a_{j-2}k^2=0$$

$$a_j=-\frac{k^2}{(m+j)^2-m^2}a_{j-2}=-\frac{k^2}{j(2m+j)}a_{j-2}$$

$$a_2=-\frac{k^2}{2(2m+2)}a_0$$

$$a_4=-\frac{k^2}{4(2m+4)}a_2=\frac{k^4}{2\cdot4\cdot(2m+2)\cdot(2m+4)}a_0$$

$$a_6=-\frac{k^2}{6(2m+6)}a_4=-\frac{k^6}{2\cdot4\cdot6\cdot(2m+2)\cdot(2m+4)\cdot(2m+6)}a_0$$

$$a_{2j}=\frac{(-1)^jk^{2j}}{2\cdot4\cdot6\cdots(2j)\cdot2^j\cdot(m+1)\cdot(m+2)\cdot(m+3)\cdots(m+j)}a_0$$

$$a_{2j}=\frac{(-1)^j}{j!\,(m+j)!}\cdot\left(\frac{k}{2}\right)^{2j}$$

$$R(\rho)=A\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\,(m+j)!}\cdot\left(\frac{k\rho}{2}\right)^{2j}+BY_m(k\rho)=AJ_m(k\rho)+BY_m(k\rho)$$

$$R(\rho)=AJ_m(k\rho),R(0)=0\to m>0,R(r)=AJ_m(kr)=0\to J_m(kr)=0$$

$$\int_0^r R^2(\rho)\rho d\rho=A^2\int_0^r J_m^2(k\rho)\rho d\rho=1$$

$$A=\left[\int_0^r J_m^2(k\rho)\rho d\rho\right]^{-1/2}$$

$$Q(\phi) = C\cosh(m\phi) + D\sinh(m\phi)$$

$$Q(0) = C\cosh(0) \rightarrow C = 0$$

$$Q(2\pi) = D\sinh(2\pi m) = 0 \rightarrow m = 0$$

$$Z(z) = E\sin(lz)$$

$$Z(h) = E\sin(lh) \rightarrow lh = n\pi \rightarrow l = \frac{n\pi}{h}$$

$$Z(z) = \sqrt{\frac{2}{h}}\sin\left(\frac{n\pi z}{h}\right)$$

$$\Phi(\rho,\phi,z) = \sqrt{\frac{2}{h}}\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\left[\int_0^r J_m^2(k\rho)\rho d\rho\right]^{-1/2} J_0(k\rho)\sin\left(\frac{n\pi z}{h}\right)$$

c. Spherical polar coordinates

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$x = r\sin\vartheta\cos\varphi$$

$$y = \sin\vartheta\sin\varphi$$

$$z = r\cos\vartheta$$

$$dx = \sin\vartheta\cos\varphi\, dr + r\cos\vartheta\cos\varphi\, d\vartheta - r\sin\vartheta\sin\varphi\, d\varphi$$

$$dy = \sin\vartheta\sin\varphi\, dr + r\cos\vartheta\sin\varphi d\vartheta + r\sin\vartheta\cos\varphi\, d\varphi$$

$$dz = \cos\vartheta\, dr - r\sin\vartheta\, d\vartheta$$

$$\begin{aligned}(dx)^2 = {} & (\sin\vartheta)^2(\cos\varphi)^2(dr)^2 + r^2(\cos\vartheta)^2(\cos\varphi)^2(d\vartheta)^2 + r^2(\sin\vartheta)^2(\sin\varphi)^2(d\varphi)^2 \\ & + 2r\sin\vartheta\cos\vartheta(\cos\varphi)^2 dr d\vartheta - 2r(\sin\vartheta)^2\cos\varphi\sin\varphi\, dr d\varphi \\ & - 2r^2\cos\vartheta\sin\vartheta\cos\varphi\sin\varphi d\vartheta d\varphi\end{aligned}$$

$$\begin{aligned}(dy)^2 = {} & (\sin\vartheta)^2(\sin\varphi)^2(dr)^2 + r^2(\cos\vartheta)^2(\sin\varphi)^2(d\vartheta)^2 + r^2(\sin\vartheta)^2(\cos\varphi)^2(d\varphi)^2 \\ & + 2r\sin\vartheta\cos\vartheta(\sin\varphi)^2 dr d\vartheta + 2r(\sin\vartheta)^2\cos\varphi\sin\varphi\, dr d\varphi \\ & + 2r^2\cos\vartheta\sin\vartheta\cos\varphi\sin\varphi d\vartheta d\varphi\end{aligned}$$

$$(dz)^2 = (\cos\vartheta)^2(dr)^2 + r^2(\sin\vartheta)^2(d\vartheta)^2 - 2r\cos\vartheta\sin\vartheta\, dr d\vartheta$$

$$\begin{aligned}(ds)^2 = {} & (dx)^2 + (dy)^2 + (dz)^2 \\ & = (\sin\vartheta)^2[(\cos\varphi)^2 + (\sin\varphi)^2](dr)^2 + (\cos\vartheta)^2(dr)^2 \\ & + r^2(\cos\vartheta)^2[(\cos\varphi)^2 + (\sin\varphi)^2](d\vartheta)^2 + r^2(\sin\vartheta)^2(d\vartheta)^2 \\ & + r^2(\sin\vartheta)^2[(\sin\varphi)^2 + (\cos\varphi)^2](d\varphi)^2 = (dr)^2 + r^2(d\vartheta)^2 + r^2(\sin\vartheta)^2(d\varphi)^2\end{aligned}$$

$$h_r = 1, h_\vartheta = r, h_\varphi = r\sin\vartheta$$

$$\nabla^2 = \frac{1}{h_r h_\vartheta h_\varphi}\left[\frac{\partial}{\partial r}\left(\frac{h_\vartheta h_\varphi}{h_r}\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial\vartheta}\left(\frac{h_r h_\varphi}{h_\vartheta}\frac{\partial}{\partial\vartheta}\right) + \frac{\partial}{\partial\varphi}\left(\frac{h_r h_\vartheta}{h_\varphi}\frac{\partial}{\partial\varphi}\right)\right]$$
$$= \frac{1}{r^2\sin\vartheta}\left[\frac{\partial}{\partial r}\left(r^2\sin\vartheta\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial\vartheta}\left(\frac{r\sin\vartheta}{r}\frac{\partial}{\partial\vartheta}\right) + \frac{\partial}{\partial\varphi}\left(\frac{r}{r\sin\vartheta}\frac{\partial}{\partial\varphi}\right)\right]$$
$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + \frac{1}{r^2(\sin\vartheta)^2}\frac{\partial^2}{\partial\varphi^2}$$

$$\nabla^2\Phi(r,\vartheta,\varphi) = 0$$

$$\Phi(r,\vartheta,\varphi) = R(r)S(\vartheta,\varphi)$$

$$\Phi(0,-\pi,0) = \Phi(a,\pi,2\pi) = 0, R(0) = R(a) = 0, S(-\pi,0) = \mathrm{S}(\pi,2\pi) = 0$$

$$\frac{1}{Rr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{Sr^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial S}{\partial\vartheta}\right) + \frac{1}{Sr^2(\sin\vartheta)^2}\frac{\partial^2 S}{\partial\varphi^2} = 0$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{S\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial S}{\partial\vartheta}\right) - \frac{1}{S(\sin\vartheta)^2}\frac{\partial^2 S}{\partial\varphi^2} = k^2$$

$$r^2R''(r) + 2rR'(r) - k^2R(r) = 0$$

$$\sin\vartheta\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial S}{\partial\vartheta}\right) + \frac{\partial^2 S}{\partial\varphi^2} + k^2(\sin\vartheta)^2S = 0$$

$$S(\vartheta,\varphi) = T(\vartheta)U(\varphi)$$

$$\frac{\sin\vartheta}{T}\frac{d}{d\vartheta}\left(\sin\vartheta\frac{dT}{d\vartheta}\right) + \frac{1}{U}\frac{d^2U}{d\varphi^2} + k^2(\sin\vartheta)^2 = 0$$

$$\frac{\sin\vartheta}{T}\frac{d}{d\vartheta}\left(\sin\vartheta\frac{dT}{d\vartheta}\right) + k^2(\sin\vartheta)^2 = -\frac{1}{U}\frac{d^2U}{d\varphi^2} = m^2$$

$$\sin\vartheta\frac{d}{d\vartheta}\left(\sin\vartheta\frac{dT}{d\vartheta}\right) + k^2(\sin\vartheta)^2T - m^2T = 0$$

$$U''(\varphi) + m^2U(\varphi) = 0$$

$$k^2 = l(l+1)$$

$$\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\frac{dT}{d\vartheta}\right) + l(l+1)T - \frac{m^2}{(\sin\vartheta)^2}T = 0$$

$$x = \cos\vartheta, y(x) = T(\vartheta)$$

$$\vartheta = \cos^{-1}x$$

$$\frac{dT}{d\vartheta} = \frac{dx}{d\vartheta}\frac{dT}{dx} = -\sin\vartheta\frac{dT}{dx}$$

$$\frac{d^2T}{d\vartheta^2} = \left(\frac{dx}{d\vartheta}\right)^2\frac{d^2T}{dx^2} + 2\frac{d^2T}{d\vartheta^2}\frac{dT}{dx} = (\sin\vartheta)^2\frac{d^2T}{dx^2} - 2\cos x\frac{dT}{dx} = (1-x^2)\frac{d^2T}{dx^2} - 2x\frac{dT}{dx}$$

$$(1-x^2)y''(x)-2xy'(x)+\left[l(l+1)-\frac{m^2}{1-x^2}\right]y(x)=0$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right]+\left[l(l+1)-\frac{m^2}{1-x^2}\right]y(x)=0$$

$$y(x)=P_l^m(x)$$

$$P_l^m(x)=\frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l=\frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}\sum_{n=0}^{l}\binom{l}{n}x^{2l}(-1)^{l-n}$$

$$U(\varphi)=B\sin(m\varphi)$$

$$A\sin(2\pi m)=0\rightarrow m\in\{1,2,\cdots\}$$

$$R(r)=\sqrt{\frac{\pi}{2\mathrm{r}}}J_{l+1/2}(r)$$

$$\Phi(r,\vartheta,\varphi)=A_{l,m}\sqrt{\frac{\pi}{2\mathrm{r}}}\sum_{l=0}^{\infty}\sum_{m=1}^{\infty}J_{l+1/2}(r)\,P_l^m(\cos\vartheta)\sin(m\varphi)$$

d. Parabolic cylindrical coordinates

$$x=\xi^2-\eta^2$$

$$y=2\xi\eta$$

$$z=z$$

$$dx=2(\xi d\xi-\eta d\eta)$$

$$dy=2(\eta d\xi+\xi d\eta)$$

$$dz=dz$$

$$\begin{aligned}ds^2=dx^2+dy^2+dz^2&=4\xi^2d\xi^2+4\eta^2d\eta^2-8\xi\eta d\xi d\eta+4\eta^2d\xi^2+4\xi^2d\eta^2+8\xi\eta d\xi d\eta+dz^2\\&=4(\xi^2+\eta^2)(d\xi^2+d\eta^2)+dz^2\end{aligned}$$

$$ds^2=g_{\xi\xi}d\xi^2+g_{\eta\eta}d\eta^2+g_{zz}dz^2$$

$$h_\xi=h_\eta=2\sqrt{\xi^2+\eta^2},h_z=1$$

$$\begin{aligned}\nabla^2&=\frac{1}{h_\xi h_\eta h_z}\left[\frac{\partial}{\partial\xi}\left(\frac{h_\eta h_z}{h_\xi}\frac{\partial}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{h_\xi h_z}{h_\eta}\frac{\partial}{\partial\eta}\right)+\frac{\partial}{\partial z}\left(\frac{h_\xi h_\eta}{h_z}\frac{\partial}{\partial z}\right)\right]\\&=\frac{1}{4}\left(\frac{1}{\xi^2+\eta^2}\right)\left[\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}+4(\xi^2+\eta^2)\frac{\partial^2}{\partial z^2}\right]=\frac{1}{4}\left(\frac{1}{\xi^2+\eta^2}\right)\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)+\frac{\partial^2}{\partial z^2}\end{aligned}$$

$$\nabla^2\Phi(x,y,z)=0$$

$$\Phi(\xi,\eta,z)=Q(x,y)Z(z)$$

$$\frac{1}{4}\left(\frac{1}{\xi^2+\eta^2}\right)\frac{1}{Q}\left(\frac{\partial^2 Q}{\partial \xi^2}+\frac{\partial^2 Q}{\partial \eta^2}\right)+\frac{1}{Z}\frac{d^2 Z}{dz^2}=-k^2$$

$$\frac{1}{4}\left(\frac{1}{\xi^2+\eta^2}\right)\frac{1}{Q}\left(\frac{\partial^2 Q}{\partial \xi^2}+\frac{\partial^2 Q}{\partial \eta^2}\right)+k^2=0$$

$$\frac{1}{Z}\frac{d^2 Z}{dz^2}+k^2=0$$

$$\left(\frac{1}{\xi^2+\eta^2}\right)\left(\frac{\partial^2 Q}{\partial \xi^2}+\frac{\partial^2 Q}{\partial \eta^2}\right)+4k^2 Q=0$$

$$Z''(z)+k^2 Z(z)=0$$

$$Z(z)=A\sin(kz)$$

$$\xi=\rho\cos\zeta$$

$$\eta=\rho\sin\zeta$$

$$\rho^2=\xi^2+\eta^2$$

$$\rho=\sqrt{\xi^2+\eta^2}$$

$$\zeta=\tan^{-1}\left(\frac{\eta}{\xi}\right)$$

$$d\xi=\cos\zeta\, d\rho-\rho\sin\zeta\, d\zeta$$
$$d\eta=\sin\zeta\, d\rho+\rho\cos\zeta\, d\zeta$$

$$d\xi^2=(\cos\zeta)^2 d\rho^2+\rho^2(\sin\zeta)^2 d\zeta^2$$

$$d\eta^2=(\sin\zeta)^2 d\rho^2+\rho^2(\cos\zeta)^2 d\zeta^2$$

$$ds^2=d\xi^2+d\eta^2=d\rho^2+\rho^2 d\zeta^2$$

$$g_{\rho\rho}=1, g_{\zeta\zeta}=\rho^2$$

$$\nabla^2=\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial\alpha^i}\left(\sqrt{|g|}g^{ij}\frac{\partial}{\partial\alpha^j}\right)$$

$$|g|=\begin{vmatrix}1 & 0\\ 0 & \rho^2\end{vmatrix}=\rho^2$$

$$\nabla^2=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho}\frac{\partial}{\partial\zeta}\left(\frac{\rho}{\rho^2}\frac{\partial}{\partial\zeta}\right)=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\zeta^2}$$

$$\frac{1}{\rho^2}\nabla^2 Q(\rho,\zeta)+4k^2 Q(\rho,\zeta)=0$$

$$Q(\rho,\zeta)=R(\rho)S(\zeta)$$

$$\frac{1}{\rho^3 R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\frac{1}{\rho^4 S}\frac{d^2 S}{d\zeta^2}+4k^2=0$$

$$\rho \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + 4k^2\rho^4 = -\frac{1}{S}\frac{d^2S}{d\zeta^2} = l^2$$

$$\rho^2 R''(\rho) + \rho R'(\rho) + (4k^2\rho^4 - l^2)R(\rho) = 0$$

$$S''(\zeta) + l^2 S(\zeta) = 0$$

$$S = B\sin(l\zeta)$$

$$R(\rho) = \sum_{j=0}^{\infty} a_j \rho^{\alpha+j}$$

$$R'(\rho) = \sum_{j=0}^{\infty} (\alpha+j) a_j \rho^{\alpha+j-1}$$

$$R''(\rho) = \sum_{j=0}^{\infty} (\alpha+j)(\alpha+j-1) a_j \rho^{\alpha+j-2}$$

$$\sum_{j=0}^{\infty} (\alpha+j)(\alpha+j-1) a_j \rho^{\alpha+j} + \sum_{j=0}^{\infty} (\alpha+j) a_j \rho^{\alpha+j} + \sum_{j=0}^{\infty} 4k^2 a_j \rho^{\alpha+j+4} - \sum_{j=0}^{\infty} l^2 a_j \rho^{\alpha+j} = 0$$

$$a_0[\alpha(\alpha-1) + \alpha - l^2] = a_0(\alpha^2 - l^2) = 0 \rightarrow \alpha = l$$

$$[l(l+1) + (l+1) - l^2]a_1 = (l^2 + 2l + 1 - l^2)a_1 = 0 \rightarrow a_1 = 0$$

$$[(l+1)(l+2) + (l+2) - l^2]a_2 = 0 \rightarrow a_2 = 0$$

$$[(l+2)(l+3) + (l+3) - l^2]a_3 = 0 \rightarrow a_3 = 0$$

$$[(l+3)(l+4) + (l+4) - l^2]a_4 + 4k^2 a_0 = (8l+16)a_4 + 4k^2 a_0 = 0$$

$$a_4 = -\frac{k^2}{2l+4}a_0$$

$$[(l+7)(l+8) + (l+8) - l^2]a_8 + 4k^2 a_4 = (16l+64)a_8 + 4k^2 a_4 = 0$$

$$a_8 = -\frac{k^2}{4l+16}a_4 = \frac{k^4}{(2l+4)(4l+16)}a_0$$

$$[(l+11)(l+12) + (l+12) - l^2]a_{12} + 4k^2 a_8 = (24l+144)a_8 + 4k^2 a_4 = 0$$

$$a_{12} = -\frac{k^2}{6l+36}a_8 = -\frac{k^6}{(2l+4)(4l+16)(6l+36)}a_0 = -\frac{k^6}{2\cdot 4\cdot 6\cdot (l+2)(l+4)(l+6)}a_0$$

$$a_{4j} = \frac{(-1)^j k^j}{2\cdot 4\cdot 6\cdots j\cdot (l+2)(l+4)(l+6)\cdots(l+j)}a_0 = \frac{(-1)^j k^j}{j!!\,(l+j)!!}a_0$$

$$R(\rho) = \sum_{j=0}^{\infty} \frac{(-1)^j k^j}{j!!\,(l+j)!!} \rho^{4j}$$

$$Q(\rho,\zeta) = A_{jl} \sum_{j=0}^{\infty} \sum_{l=1}^{\infty} \frac{(-1)^j k^j}{j!!\,(l+j)!!} \rho^{4j} \sin(l\zeta)$$