The Schrödinger Equation for the Hydrogen-Like Atom

$$-\frac{h^2}{8\pi^2\mu}\nabla^2\psi - \frac{Ze^2}{r}\psi = E\psi$$

$$\nabla^2\psi + \frac{8\pi^2\mu Ze^2}{h^2 r}\psi + \frac{8\pi^2\mu E}{h^2}\psi = 0$$

$$\frac{1}{\mu} = \frac{1}{m} + \frac{1}{M} = \frac{M+m}{mM}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$x = r\sin\vartheta\cos\varphi$$

$$y = \ \sin\vartheta\sin\varphi$$

$$z = r\cos\vartheta$$

$$dx = \sin\vartheta\cos\varphi\, dr + r\cos\vartheta\cos\varphi\, d\vartheta - r\sin\vartheta\sin\varphi\, d\varphi$$

$$dy = \sin\vartheta\sin\varphi\, dr + r\cos\vartheta\sin\varphi d\vartheta + r\sin\vartheta\cos\varphi\, d\varphi$$

$$dz = \cos\vartheta\, dr - r\sin\vartheta\, d\vartheta$$

$$\begin{aligned}(dx)^2 = (\sin\vartheta)^2(\cos\varphi)^2(dr)^2 + r^2(\cos\vartheta)^2(\cos\varphi)^2(d\vartheta)^2 + r^2(\sin\vartheta)^2(\sin\varphi)^2(d\varphi)^2 \\ + 2r\sin\vartheta\cos\vartheta(\cos\varphi)^2 dr d\vartheta - 2r(\sin\vartheta)^2\cos\varphi\sin\varphi\, dr d\varphi \\ - 2r^2\cos\vartheta\sin\vartheta\cos\varphi\sin\varphi d\vartheta d\varphi\end{aligned}$$

$$\begin{aligned}(dy)^2 = (\sin\vartheta)^2(\sin\varphi)^2(dr)^2 + r^2(\cos\vartheta)^2(\sin\varphi)^2(d\vartheta)^2 + r^2(\sin\vartheta)^2(\cos\varphi)^2(d\varphi)^2 \\ + 2r\sin\vartheta\cos\vartheta(\sin\varphi)^2 dr d\vartheta + 2r(\sin\vartheta)^2\cos\varphi\sin\varphi\, dr d\varphi \\ + 2r^2\cos\vartheta\sin\vartheta\cos\varphi\sin\varphi d\vartheta d\varphi\end{aligned}$$

$$(dz)^2 = (\cos\vartheta)^2(dr)^2 + r^2(\sin\vartheta)^2(d\vartheta)^2 - 2r\cos\vartheta\sin\vartheta\, dr d\vartheta$$

$$\begin{aligned}(ds)^2 &= (dx)^2 + (dy)^2 + (dz)^2 \\ &= (\sin\vartheta)^2[(\cos\varphi)^2 + (\sin\varphi)^2](dr)^2 + (\cos\vartheta)^2(dr)^2 \\ &+ r^2(\cos\vartheta)^2[(\cos\varphi)^2 + (\sin\varphi)^2](d\vartheta)^2 + r^2(\sin\vartheta)^2(d\vartheta)^2 \\ &+ r^2(\sin\vartheta)^2[(\sin\varphi)^2 + (\cos\varphi)^2](d\varphi)^2 = (dr)^2 + r^2(d\vartheta)^2 + r^2(\sin\vartheta)^2(d\varphi)^2\end{aligned}$$

$$h_r = 1, h_\vartheta = r, h_\varphi = r\sin\vartheta$$

$$\nabla^2 = \frac{1}{h_r h_\vartheta h_\varphi}\left[\frac{\partial}{\partial r}\left(\frac{h_\vartheta h_\varphi}{h_r}\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial \vartheta}\left(\frac{h_r h_\varphi}{h_\vartheta}\frac{\partial}{\partial \vartheta}\right) + \frac{\partial}{\partial \varphi}\left(\frac{h_r h_\vartheta}{h_\varphi}\frac{\partial}{\partial \varphi}\right)\right]$$

$$= \frac{1}{r^2 \sin\vartheta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\vartheta \frac{\partial}{\partial r}\right) + \frac{\partial}{\partial \vartheta}\left(\frac{r \sin\vartheta}{r}\frac{\partial}{\partial \vartheta}\right) + \frac{\partial}{\partial \varphi}\left(\frac{r}{r \sin\vartheta}\frac{\partial}{\partial \varphi}\right)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial}{\partial \vartheta}\right) + \frac{1}{r^2 (\sin\vartheta)^2}\frac{\partial^2}{\partial \varphi^2}$$

$$\psi(r,\vartheta,\varphi) = R(r)Y(\vartheta,\varphi)$$

$$\frac{Y}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{R}{r^2 \sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial Y}{\partial \vartheta}\right) + \frac{R}{r^2 (\sin\vartheta)^2}\frac{\partial^2 Y}{\partial \varphi^2} + \frac{8\pi^2 \mu Z e^2}{h^2 r} RY + \frac{8\pi^2 \mu E}{h^2} RY = 0$$

$$\frac{1}{R r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{1}{Y r^2 \sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial Y}{\partial \vartheta}\right) + \frac{1}{Y r^2 (\sin\vartheta)^2}\frac{\partial^2 Y}{\partial \varphi^2} + \frac{8\pi^2 \mu Z e^2}{h^2 r} + \frac{8\pi^2 \mu E}{h^2} = 0$$

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{1}{Y \sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial Y}{\partial \vartheta}\right) + \frac{1}{Y (\sin\vartheta)^2}\frac{\partial^2 Y}{\partial \varphi^2} + \frac{8\pi^2 \mu Z e^2}{h^2} r + \frac{8\pi^2 \mu E}{h^2} r^2 = 0$$

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{8\pi^2 \mu Z e^2}{h^2} r + \frac{8\pi^2 \mu E}{h^2} r^2 = -\frac{1}{Y \sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial Y}{\partial \vartheta}\right) - \frac{1}{Y (\sin\vartheta)^2}\frac{\partial^2 Y}{\partial \varphi^2} = \lambda^2$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{8\pi^2 \mu Z e^2}{h^2} r + \frac{8\pi^2 \mu E}{h^2} r^2 - \lambda^2 = 0$$

$$-\frac{1}{Y \sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial Y}{\partial \vartheta}\right) - \frac{1}{Y (\sin\vartheta)^2}\frac{\partial^2 Y}{\partial \varphi^2} - \lambda^2 = 0$$

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{8\pi^2 \mu Z e^2}{h^2} rR + \frac{8\pi^2 \mu E}{h^2} r^2 R - \lambda^2 R = 0$$

$$-\frac{1}{\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial Y}{\partial \vartheta}\right) - \frac{1}{(\sin\vartheta)^2}\frac{\partial^2 Y}{\partial \varphi^2} - \lambda^2 Y = 0$$

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) - \left(\lambda^2 - \frac{8\pi^2 \mu Z e^2}{h^2} r - \frac{\mu E}{h^2} r^2\right) R = 0$$

$$Y(\vartheta,\varphi) = \Theta(\vartheta)\Phi(\varphi)$$

$$-\frac{\Phi}{\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial \Theta}{\partial \vartheta}\right) - \frac{\Theta}{(\sin\vartheta)^2}\frac{\partial^2 \Phi}{\partial \varphi^2} - \lambda^2 \Theta\Phi = 0$$

$$-\frac{1}{\Theta \sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial \Theta}{\partial \vartheta}\right) - \frac{1}{\Phi (\sin\vartheta)^2}\frac{\partial^2 \Phi}{\partial \varphi^2} - \lambda^2 = 0$$

$$-\frac{\sin\vartheta}{\Theta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial \Theta}{\partial \vartheta}\right) - \frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \varphi^2} - \lambda^2 (\sin\vartheta)^2 = 0$$

$$-\frac{\sin\vartheta}{\Theta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial \Theta}{\partial \vartheta}\right) - \lambda^2 (\sin\vartheta)^2 = -\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \varphi^2} = \nu^2$$

$$\sin\vartheta\frac{d}{d\vartheta}\left(\sin\vartheta\frac{\mathrm{d}\Theta}{d\vartheta}\right)+\lambda^2(\sin\vartheta)^2\Theta-\nu^2\Theta=0$$

$$\frac{d^2\Phi}{d\varphi^2}+\nu^2\Phi=0$$

$$m=\nu$$

$$\Phi(\varphi)=Ae^{im\varphi}+Be^{-im\varphi},m\neq 0$$

$$\Phi(\varphi)=A+B\varphi,m=0$$

$$\Phi_m(\varphi)=\frac{1}{\sqrt{2\pi}}e^{im\varphi}$$

$$\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\frac{\mathrm{d}\Theta}{d\vartheta}\right)+\lambda^2\Theta-\frac{m^2\Theta}{(\sin\vartheta)^2}=0$$

$$x=\cos\vartheta$$

$$\frac{\mathrm{d}\Theta}{d\vartheta}=\frac{dx}{d\vartheta}\frac{\mathrm{d}\Theta}{dx}=-\sin\vartheta\frac{\mathrm{d}\Theta}{dx}$$

$$\frac{\mathrm{d}}{d\vartheta}=\frac{dx}{d\vartheta}\frac{\mathrm{d}}{dx}=-\sin\vartheta\frac{\mathrm{d}}{dx}$$

$$\frac{\mathrm{d}}{dx}\left[(1-x^2)\frac{\mathrm{d}\Theta}{dx}\right]+\left[\lambda^2-\frac{m^2}{(1-x^2)}\right]\Theta=0$$

$$Y_{lm}(\vartheta,\varphi)=N_{lm}P_l^m(\cos\vartheta)\Phi_m(\varphi)=\epsilon\sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)}{(l+|m|)}}P_l^m(\cos\vartheta)\Phi_m(\varphi)$$

$$\epsilon=(-1)^m\forall m>0,\epsilon=1\forall m\leq 0$$

$$Y_{0,0}=\frac{1}{\sqrt{4\pi}}$$

$$Y_{1,0}=\sqrt{\frac{3}{4\pi}}\cos\vartheta$$

$$Y_{1,\pm 1}=\mp\sqrt{\frac{3}{8\pi}}\sin\vartheta\, e^{\pm i\varphi}$$

$$Y_{2,0}=\sqrt{\frac{5}{16\pi}}[3(\cos\vartheta)^2-1]$$

$$Y_{2,\pm 1}=\mp\sqrt{\frac{15}{8\pi}}\sin\vartheta\cos\vartheta\, e^{\pm i\varphi}$$

$$Y_{2,\pm 2} = \sqrt{\frac{15}{32\pi}}(\sin\vartheta)^2 e^{\pm 2i\varphi}$$

$$\frac{h^2}{4\pi^2\mu}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-\left(\frac{h^2\lambda^2}{4\pi^2\mu r^2}-\frac{Ze^2}{r}-E\right)R = 0$$

$$\rho = \alpha r$$

$$\frac{d}{dr} = \frac{d\rho}{dr}\frac{d}{d\rho} = \alpha\frac{d}{d\rho}$$

$$\frac{h^2}{8\pi^2\mu}\frac{\alpha^2}{\rho^2}\alpha\frac{d}{d\rho}\left(\frac{\rho^2}{\alpha^2}\alpha\frac{dR}{d\rho}\right)-\left(\frac{h^2\lambda^2\alpha^2}{8\pi^2\mu\rho^2}-\frac{Ze^2\alpha}{\rho}-E\right)R = 0$$

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)-\left(\frac{\lambda^2}{\rho^2}-\frac{8\pi^2\mu Ze^2}{h^2\alpha\rho}-\frac{8\pi^2\mu E}{h^2\alpha^2}\right)R = 0$$

$$-\frac{1}{4} = -\frac{8\pi^2\mu E}{h^2\alpha^2}$$

$$\alpha^2 = \frac{8\pi^2\mu|E|}{h^2}$$

$$\beta = \frac{8\pi^2\mu Ze^2}{h^2\alpha}$$

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)+\left(\frac{\beta}{\rho}-\frac{\lambda^2}{\rho^2}-\frac{1}{4}\right)R = 0$$

$$R(\rho) = e^{-a\rho}F(\rho)$$

$$\frac{dR}{d\rho} = e^{-a\rho}F'(\rho) - ae^{-a\rho}F(\rho)$$

$$\frac{d^2R}{d\rho^2} = e^{-a\rho}F''(\rho) - 2ae^{-a\rho}F'(\rho) + a^2e^{-a\rho}F(\rho)$$

$$\frac{d^2R}{d\rho^2}+\frac{2}{\rho}\frac{dR}{dr}+\left(\frac{\beta}{\rho}-\frac{\lambda^2}{\rho^2}-\frac{1}{4}\right)R = 0$$

$$e^{-a\rho}F''(\rho) - 2ae^{-a\rho}F'(\rho) + a^2e^{-a\rho}F(\rho) + \frac{2}{\rho}e^{-a\rho}F'(\rho) - \frac{2}{\rho}ae^{-a\rho}F(\rho) + \left(\frac{\beta}{\rho}-\frac{\lambda^2}{\rho^2}-\frac{1}{4}\right)e^{-a\rho}F(\rho) = 0$$

$$F''(\rho) - 2aF'(\rho) + a^2F(\rho) + \frac{2}{\rho}F'(\rho) - \frac{2}{\rho}aF(\rho) + \left(\frac{\beta}{\rho}-\frac{\lambda^2}{\rho^2}\right)F(\rho) = 0$$

$$F''(\rho) + \left(\frac{2}{\rho}-1\right)F'(\rho) + \left(\frac{\beta}{\rho}-\frac{\lambda^2}{\rho^2}\right)F(\rho) = 0, a = 1/2$$

$$F(\rho) = \rho^s L(\rho)$$

$$F'(\rho) = \rho^s L'(\rho) + s\rho^{s-1}L(\rho)$$

$$F''(\rho) = \rho^s L''(\rho) + 2s\rho^{s-1}L'(\rho) + s(s-1)\rho^{s-2}L(\rho)$$

$$\rho^s L''(\rho) + 2s\rho^{s-1}L'(\rho) + s(s-1)\rho^{s-2}L(\rho) + \left(\frac{2}{\rho} - 1\right)[\rho^s L'(\rho) + s\rho^{s-1}L(\rho)] + \left(\frac{\beta}{\rho} - \frac{\lambda^2}{\rho^2}\right)\rho^s L(\rho) = 0$$

$$\rho^2 L'' + 2s\rho L' + s(s-1)L + (2\rho - \rho^2)(L' + s\rho L) + (\beta\rho - \lambda^2)L = 0$$

$$\rho^2 L'' + \rho[2(s+1) - \rho]L' + [\rho(\beta - s - 1) + s(s+1) - \lambda^2]L = 0$$

$$\rho = 0 \to s(s+1) - \lambda^2 = 0$$

$$\lambda^2 = l(l+1)$$

$$s = l, s = -(l+1)$$

$$\rho^2 L'' + \rho[2(l+1) - \rho]L' + [\rho(\beta - l - 1) + l(l+1) - l(l+1)]L = 0$$

$$\rho L'' + [2(l+1) - \rho]L' + (\beta - l - 1)L = 0$$

$$L(\rho) = \sum_{k=0}^{\infty} a_k \rho^k$$

$$k(k-1)a_k\rho^{k-1} + 2(l+1)ka_k\rho^{k-1} - ka_k\rho^k + (\beta - l - 1)a_k\rho^k = 0$$

$$k(k+1)a_{k+1}\rho^k + 2(l+1)(k+1)a_{k+1}\rho^k + (\beta - l - 1 - k)a_k\rho^k = 0$$

$$a_{k+1} = \frac{k + l + 1 - \beta}{(k+1)(k+2l+2)}a_k$$