Number Representation by James Pate Williams, Jr.

Consider the base 10 number that is also a positive integer, 1234, which is really shorthand for the finite series expansion:

$$1{,}234 = 1 \times 10^3 + 2 \times 10^2 + 3 \times 10^1 + 4 \times 10^0$$

Similarly, we obtain for 1,234,567:

$$1{,}234{,}567 = 1 \times 10^6 + 2 \times 10^5 + 3 \times 10^4 + 4 \times 10^3 + 5 \times 10^2 + 6 \times 10^1 + 7 \times 10^0$$

Now imagine the number system that is created using the Arabic numerals 0 and 1, that is, the binary number system. The number represented by 10010110001101 is an abbreviation of the expansion:

$$\begin{aligned} 10010110001101 &= 1 \times 2^{13} + 0 \times 2^{12} + 0 \times 2^{11} + 1 \times 2^{10} + 0 \times 2^9 + 1 \times 2^8 + 1 \times 2^7 + 0 \times 2^6 \\ &+ 0 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \\ &= 8192 + 1024 + 256 + 128 + 8 + 4 + 1 = 9613 \end{aligned}$$

Finally, suppose we are working in base, also known as radix, sixteen or the hexadecimal number system. The allowed digits are 0-9 and A-F, where A=10, B=11, C=12, D=13, E=14, and F=15. The number ABCD is shorthand for:

$$ABCD = A \times 16^3 + B \times 16^2 + C \times 16^1 + D \times 16^0 = 10 \times 4096 + 11 \times 256 + 12 \times 16 + 13 = 43{,}981$$

Conversion from base 16, hexadecimal, to base 2, binary is very easy. A hexadecimal number is readily converted using the table:

Base-16	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
Base-2	0000	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	1110	1111

For example, take ABCD, which is:

$$ABCD_{16} = 43{,}981_{10} = 1010_2 1011_2 1100_2 1101_2 = 1010101111001101_2$$

We sometimes write the base or radix as a subscript.

Boolean algebra

Boolean algebra, which was developed by George Boole in 1854, uses the binary number system consisting of the digits, also known as bits which are a contraction of **b**inary dig**its**, 0 and 1. It has three basic operations known as conjunction (and), disjunction (or), and negation (complementation). The equation below uses logical conjunction:

$$z = x \wedge y$$

The preceding equation represents the logical conjunction (and) of x and y. Sometimes the preceding equation is written more confusingly as:

$$z = xy = x \cdot y$$

Similarly, the equation below uses logical disjunction:

$$z = x \vee y$$

The preceding equation represents the logical disjunction (or) of x and y. Again, using the normal algebraic notation of the plus or addition sign, the preceding equation can be written:

$$z = x + y$$

Finally, we show an equation using the negation (not) operator of Boolean algebra:

$$z = \neg x$$

Now for the truth tables for the three fundamental operations of and, or, and not:

x	y	x*y
0	0	0
0	1	0
1	0	0
1	1	1

x	y	x+y
0	0	0
0	1	1
1	0	1
1	1	1

x	$\neg x$
0	1
1	0

Suppose we want to determine the result of the bitwise logical conjunctions of AB with the mask CD then we obtain:

$$AB \wedge CD = 1010_2 1011_2 \wedge 1100_2 1101_2 = 1000_2 1001_2 = 89_{16}$$

Since we have:

$$A \wedge B = 1010_2 \wedge 1100_2 = 1000_2 = 8_{16}$$

$$\wedge D = 1011_2 \wedge 1101_2 = 1001_2 = 9_{16}$$

By the same token, we find for the logical disjunction or bitwise or operations:

$$\vee CD = 1010_2 1011_2 \vee 1100_2 1101_2 = 1110_2 1111_2 = E_{16} F_{16}$$

Due to the facts:

$$A \vee C = 1010_2 \vee 1100_2 = 1110_2 = E_{16}$$

$$B \vee D = 1011_2 \vee 1101_2 = 1111_2 = F_{16}$$

A bitwise logical negation or complementation example is shown below:

$$\neg A = \neg 1010_2 = 0101_2 = 5_{16}$$

Exercises

Instructions: show all work and only use Microsoft's Vista or higher operating system calculator in the "Programmer View" to verify your hand calculations. Exercises must be labeled as shown below and the work should be done as in the equations in the preceding sections using Microsoft's Word 2010 word processor and its associated mathematical equation editor.

1. Represent the following decimal numbers as finite series expansions.
   a. 127

   $$127 = 1 \times 10^2 + 2 \times 10^1 + 7 \times 10^0$$

   b. 1,023

   $$1{,}023 = 1 \times 10^3 + 0 \times 10^2 + 2 \times 10^1 + 3 \times 10^0$$ c.

   16,383

   $$16{,}383 = 1 \times 10^4 + 6 \times 10^3 + 3 \times 10^2 + 8 \times 10^1 + 3 \times 10^0$$ d.

   262,143

   $$262{,}143 = 2 \times 10^5 + 6 \times 10^4 + 2 \times 10^3 + 1 \times 10^2 + 4 \times 10^1 + 3 \times 10^0$$

2. Write out the finite series expansions of the hexadecimal numbers below.
   a. B9

   $$B9 = 11 \times 16^1 + 9 \times 16^0$$

   b. DF4

   $$DF4 = 13 \times 16^2 + 15 \times 16^1 + 4 \times 16^0$$

   c. 7F6E

$$7F6E = 7 \times 16^3 + 15 \times 16^2 + 6 \times 16^1 + 14 \times 16^0$$

d. 3A5D8C

$$3A5D8C = 3 \times 16^5 + 10 \times 16^4 + 5 \times 16^3 + 13 \times 16^2 + 8 \times 16^1 + 12 \times 16^0$$

3. Show the finite series expansions of the following binary numbers.

   a. 110

$$110 = 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$$

   b. 101011

$$101011 = 1 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$$

   c. 11001011

$$11001011 \quad = 1 \times 2^7 + 1 \times 2^6 + 0 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1$$

$+ 1 \times 2^0$ d. 11100101001

$$11100101001 = 1 \times 2^{10} + 1 \times 2^9 + 1 \times 2^8 + 0 \times 2^7 + 0 \times 2^6 + 1 \times 2^5 + 0 \times 2^4$$
$$+ 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$

4. Convert the numbers in Exercise 1 to hexadecimal and binary.

   a. 127

$$127 = 1 \times 10^2 + 2 \times 10^1 + 7 \times 10^0$$
$$= 1 \times 2^6 + 1 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$$
$$= 1111111 = 7F = 7 \times 16^1 + 15 \times 16^0$$

   b. 1,023

$$1{,}023 = 1 \times 10^3 + 0 \times 10^2 + 2 \times 10^1 + 3 \times 10^0$$
$$= 512 + 256 + 128 + 64 + 32 + 16 + 8 + 4 + 2 + 1 = 1111111111$$
$$= 3FF$$

   c. 16,383

$$16{,}383 = 1 \times 10^4 + 6 \times 10^3 + 3 \times 10^2 + 8 \times 10^1 + 3 \times 10^0$$
$$= 8{,}192 + 4{,}096 + 2{,}048 + 1{,}024 + 1{,}023 = 11111111111111$$
$$= 3FFF$$

   d. 262,143

$$262{,}143 = 2 \times 10^5 + 6 \times 10^4 + 2 \times 10^3 + 1 \times 10^2 + 4 \times 10^1 + 3 \times 10^0$$
$$= 131{,}072 + 65{,}536 + 32{,}768 + 16{,}384 + 16{,}383 = 111111111111111111$$
$$= 3FFFF$$

5. Convert the numbers in Exercise 2 to binary and decimal.

   a. B9

$$B9 = 11 \times 16^1 + 9 \times 16^0 = 185 = 10111001$$

   b. DF4

$DF4 = 13 \times 16^2 + 15 \times 16^1 + 4 \times 16^0 = 3{,}572 = 110111110100$ c. 7F6E

$7F6E = 7 \times 16^3 + 15 \times 16^2 + 6 \times 16^1 + 14 \times 16^0 = 32{,}622 = 111111101101110$ d. 3A5D8C

$$3A5D8C = 3 \times 16^5 + 10 \times 16^4 + 5 \times 16^3 + 13 \times 16^2 + 8 \times 16^1 + 12 \times 16^0 = 3{,}825{,}036$$
$$= 1110100101110110001100$$

6. Convert the numbers in Exercise 3 to hexadecimal and decimal.
   a. 110
   $$110 = 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 6_{16} = 6_{10}$$
   b. 101011
   $$101011 = 1 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 2B_{16} = 43_{10}$$
   c. 11001011
   $$11001011 \quad = 1 \times 2^7 + 1 \times 2^6 + 0 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$$
   $$= CB_{16} = 203_{10}$$
   d. 11100101001
   $$11100101001 = 1 \times 2^{10} + 1 \times 2^9 + 1 \times 2^8 + 0 \times 2^7 + 0 \times 2^6 + 1 \times 2^5 + 0 \times 2^4$$
   $$+ 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 729_{16} = 1{,}833_{10}$$

7. What are the bitwise logical conjunctions of the following.
   a. 5D and B7
   $$5D \wedge B7 = 01011101_2 \wedge 10110111_2 = 0001_2 0101_2 = 15_{16}$$ b.
   C4 and EA
   $$C4 \wedge EA = 1100_2 0100_2 \wedge 1110_2 1010_2 = 1100_2 0000_2 = C0_{16}$$
8. What are the bitwise logical disjunctions of the bytes displayed below.
   a. F8 and 7B
   $$F8 \vee 7B = 1111_2 1000_2 \vee 0111_2 1011_2 = 1111_2 1011_2 = FB_{16}$$ b. 36
   and AD
   $$36 \vee AD = 0011_2 0110_2 \vee 1010_2 1101_2 = 1011_2 1111_2 = BF_{16}$$
9. What are the bitwise logical negations of the bytes.
   a. AC
   $$\neg AC = \neg 1010_2 1100_2 = 0101_2 0011_2 = 53_{16}$$
   b. E7
   $$\neg E7 = \neg 1110_2 0111_2 = 0001_2 1000_2 = 18_{16}$$
   c. F2
   $$\neg F2 = \neg 1111_2 0010_2 = 0000_2 1101_2 = 0D_{16}$$
   d. CD
   $$\neg CD = \neg 1100_2 1101_2 = 0011_2 0010_2 = 32_{16}$$