[Essential Mathematical Methods for Physicists - Weber and Arfken.1.1 (cofc.edu)]

Exercise 11.3.4 Neutrons (mass 1) being scattered by a nucleus of mass A (A > 1). In the center of mass system, the scattering is isotropic. Then in lab system the average cosine of the angle of deflection of the neutron is:

$$\langle\cos\psi\rangle = \frac{1}{2}\int_0^\pi \frac{A\cos\theta + 1}{(A^2 + 2A\cos\theta + 1)^{1/2}}\sin\theta d\theta.$$

Show, by expansion of the denominator that:

$$\langle\cos\psi\rangle = \frac{2}{3A}.$$

Here is my solution, James Pate Williams, Jr.,

$$f(\theta) = (A^2 + 2A\cos\theta + 1)^{-1/2}$$

$$f'(\theta) = A\sin\theta(A^2 + 2A\cos\theta + 1)^{-3/2}$$

$$f''(\theta) = A\cos\theta\,(A^2 + 2A\cos\theta + 1)^{-3/2} + 3A^2(\sin\theta)^2(A^2 + 2A\cos\theta + 1)^{-5/2}$$

$$f'''(\theta) = -A\sin\theta\,(A^2 + 2A\cos\theta + 1)^{-3/2} + 3A^2(\cos\theta)^2(A^2 + 2A\cos\theta + 1)^{-5/2} + 6A^2\sin\theta\cos\theta\,(A^2 + 2A\cos\theta + 1)^{-5/2} + 15A^3(\sin\theta)^3(A^2 + 2A\cos\theta + 1)^{-7/2}$$

$$f^{iv}(\theta) = -\mathrm{A}\cos\theta(A^2 + 2A\cos\theta + 1)^{-3/2} - 3A^2(\sin\theta)^2\,(A^2 + 2A\cos\theta + 1)^{-5/2} - 6A^2\sin\theta\cos\theta(A^2 + 2A\cos\theta + 1)^{-5/2} + 15A^3(\cos\theta)^2\,(A^2 + 2A\cos\theta + 1)^{-7/2} + 6A^2(\cos\theta)^2(A^2 + 2A\cos\theta + 1)^{-5/2} + 6A^2(\sin\theta)^2(A^2 + 2A\cos\theta + 1)^{-5/2} + 30A^3(\sin\theta)^2\cos\theta\,(A^2 + 2A\cos\theta + 1)^{-5/2} + 45A^3\sin\theta\cos\theta\,(A^2 + 2A\cos\theta + 1)^{-7/2} + 105A^4(\sin\theta)^4(A^2 + 2A\cos\theta + 1)^{-9/2}$$

$$f(0) = (A^2 + 2A + 1)^{-1/2} = (A + 1)^{-1}$$

$$f'(0) = 0$$

$$f''(0) = A(A + 1)^{-3}$$

$$f'''(0) = 3A^2(A + 1)^{-5}$$

$$f^{iv}(0) = -A(A + 1)^{-3} + 15A^3(A + 1)^{-7} + 6A^2(A + 1)^{-5}$$

$$f(\theta) = (A + 1)^{-1} + A(A + 1)^{-3}\frac{\theta^2}{2!} + 3A^2(A + 1)^{-5}\frac{\theta^3}{2!} - A(A + 1)^{-3}\frac{\theta^4}{4!} + 6A^2(A + 1)^{-5}\frac{\theta^4}{4!} + 15A^3(A + 1)^{-7}\frac{\theta^4}{4!} + \cdots$$

I sort of gave up on my Maclaurin series expansion.

$$\langle\cos\psi\rangle = \frac{1}{2}(A + 1)^{-1}\left(\int_0^\pi A\cos\theta\sin\theta d\theta + \int_0^\pi \sin\theta d\theta\right) + \cdots$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{2i\theta} = \cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta) = (\cos\theta)^2 - (\sin\theta)^2 + 2i\sin\theta\cos\theta$$

Equating the real and imaginary terms we get:

$$\cos 2\theta = (\cos\theta)^2 - (\sin\theta)^2$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\int_0^\pi A\cos\theta\sin\theta\, d\theta = \frac{A}{2}\int_0^\pi 2\sin 2\theta\, d\theta = -\frac{A}{2}[\cos 2\pi - \cos 0] = \frac{A}{2}$$

$$\int_0^\pi \sin\theta d\theta = -[\cos\pi - \cos 0] = 2$$

$$\langle\cos\psi\rangle = \frac{1}{2}(A+1)^{-1}\left(\frac{A}{2}+2\right) + \cdots$$

Using [Integral Calculator • With Steps! (integral-calculator.com)](integral-calculator.com):

$$z = \cos\theta, dz = -\sin\theta d\theta$$

$$\langle\cos\psi\rangle = \frac{1}{2}\int_{-1}^{1}\frac{Az+1}{(A^2+2Az+1)^{1/2}}dz = \left.\frac{(Az - A^2 + 2)\sqrt{2Az + A^2 + 1}}{6A}\right]_{z=-1}^{z=1}$$
$$= \frac{(A - A^2 + 2)\sqrt{2A + A^2 + 1}}{6A} - \frac{(-A - A^2 + 2)\sqrt{-2A + A^2 + 1}}{6A}$$
$$= -\frac{(-A - A^2 + 2)\sqrt{-2A + A^2 + 1}}{6A}$$

Now assume A = 2:

$$\langle\cos\psi\rangle = \frac{4}{12} = \frac{1}{3}$$

$$g(z) = (A^2 + 2Az + 1)^{-1/2}$$

$$g'(z) = -A(A^2 + 2Az + 1)^{-3/2}$$

$$g''(z) = 3A^2(A^2 + 2Az + 1)^{-5/2}$$

$$g'''(z) = -15A^3(A^2 + 2Az + 1)^{-7/2}$$

$$g^{iv}(z) = 105A^4(A^2 + 2Az + 1)^{-9/2}$$

$$g^{v}(z) = -945A^5(A^2 + 2Az + 1)^{-11/2}$$

$$g(0) = (A^2 + 1)^{-1/2}$$

$$g'(0) = -A(A^2 + 1)^{-3/2}$$

$$g''(0) = 3A^2(A^2 + 1)^{-5/2}$$

$$g'''(0) = -15A^3(A^2+1)^{-7/2}$$

$$g^{iv}(0) = 105A^4(A^2+1)^{-9/2}$$

$$g^{v}(0) = -945A^5(A^2+1)^{-11/2}$$

$$g(z) = (A^2+1)^{-1/2} - A(A^2+1)^{-3/2}z + 3(A^2+1)^{-5/2}\frac{z^2}{2} - 15A^3(A^2+1)^{-7/2}\frac{z^3}{3!} + 105A^4(A^2+1)^{-9/2}\frac{z^4}{4!} - 945A^5(A^2+1)^{-11/2}\frac{z^5}{5!} + \cdots$$

$$\langle\cos\psi\rangle = \frac{A}{2}\int_{-1}^{1} zg(z)dz + \frac{1}{2}\int_{-1}^{1} g(z)dz = A\,(A^2+1)^{-1/2} + A(A^2+1)^{-5/2} + \cdots + 2(A^2+1)^{-1/2} - 10A^3(A^2+1)^{-7/2} + \cdots$$

I leave the completion of the integral calculation to the aspiring reader.

Here are some agreeable numerical computations:

```
Enter Mass of Nucleus A: 2
Enter Integration Steps: 1024
Scattering Cos Angle: 0.333333
Scattering Angle Degrees: 70
Scattering Angle Minutes: 31
Scattering Angle Seconds: 43
2 / (3 * A) = 0.333333
Enter Mass of Nucleus A: 3
Enter Integration Steps: 1024
Scattering Cos Angle: 0.222222
Scattering Angle Degrees: 77
Scattering Angle Minutes: 9
Scattering Angle Seconds: 37
2 / (3 * A) = 0.222222
Enter Mass of Nucleus A: 4
Enter Integration Steps: 1024
Scattering Cos Angle: 0.166667
Scattering Angle Degrees: 80
Scattering Angle Minutes: 24
Scattering Angle Seconds: 21
2 / (3 * A) = 0.166667
Enter Mass of Nucleus A: 5
Enter Integration Steps: 1024
Scattering Cos Angle: 0.133333
Scattering Angle Degrees: 82
Scattering Angle Minutes: 20
Scattering Angle Seconds: 15
2 / (3 * A) = 0.133333
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Enter Mass of Nucleus A: 6
Enter Integration Steps: 1024
Scattering Cos Angle: 0.111111
Scattering Angle Degrees: 83
Scattering Angle Minutes: 37
Scattering Angle Seconds: 14
2 / (3 * A) = 0.111111
Enter Mass of Nucleus A: 7
Enter Integration Steps: 1024
Scattering Cos Angle: 0.0952381
Scattering Angle Degrees: 84
Scattering Angle Minutes: 32
Scattering Angle Seconds: 5
2 / (3 * A) = 0.0952381
Enter Mass of Nucleus A: 8
Enter Integration Steps: 1024
Scattering Cos Angle: 0.0833333
Scattering Angle Degrees: 85
Scattering Angle Minutes: 13
Scattering Angle Seconds: 11
2 / (3 * A) = 0.0833333
Enter Mass of Nucleus A: 9
Enter Integration Steps: 1024
Scattering Cos Angle: 0.0740741
Scattering Angle Degrees: 85
Scattering Angle Minutes: 45
Scattering Angle Seconds: 7
2 / (3 * A) = 0.0740741
Enter Mass of Nucleus A: 10
Enter Integration Steps: 1024
Scattering Cos Angle: 0.0666667
Scattering Angle Degrees: 86
Scattering Angle Minutes: 10
Scattering Angle Seconds: 38
2 / (3 * A) = 0.0666667
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