Quantum Mechanics by Leonard I. Schiff 1968

Problems from Chapter 6 Matrix Formulation of Quantum Mechanics

1. Assume that any Hermitian matrix can be diagonalized by a unitary matrix. From this, show that the necessary and sufficient condition that two Hermitian matrices can be diagonalized by the same unitary transformation is that they commute.

Definition of a unitary matrix page 152:

$$\widehat{U} = U^{-1}, U\widehat{U} = I, \widehat{U}U = I$$

Definition of transformation of a square matrix page 152:

$$SAS^{-1} = A'$$

Diagonal square matrix page 152:

$$A'_{kl} = A'_k \delta_{kl,} \delta_{kl} = 1 \; for \; k = l \; else \; \delta_{kl} = 0 \; for \; k \neq l$$

Let

$$UAU^{-1} = A'$$

$$UBU^{-1} = B'$$

The commutator of A and B is given by:

$$[A, B] = AB - BA$$

Multiply the commutator by the Hermitian matrix U and its inverse:

$$U[A, B]U^{-1} = UABU^{-1} - UBAU^{-1} = A' - B'$$

Suppose AB and BA are diagonal then [A, B] must be zero.

2. Show that a nonsingular matrix of finite rank must be square. Also show in this case the following equation

$$AA^{-1} = I$$

Implies the equation:

$$A^{-1}A = I.$$

A nonsingular matrix of finite rank must be square since

$$\det(A) \neq 0$$

Suppose the matrix is:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & b \\ c & a \end{bmatrix}$$

$$AA^{-1} = \frac{1}{ad - bc}\begin{bmatrix} a & -b \\ -c & d \end{bmatrix}\begin{bmatrix} d & b \\ c & a \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} ad - bc & ab - ba \\ -cd + dc & -bc + ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Likewise

$$A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det(A) = ad - bc \neq 0$$

3. Show that the following equation is true:

$$\det(e^{A}) = e^{tr(A)}$$

Determinants have the following properties:

$$\det(AB) = \det(A)\det(B)$$

$$\det(A^2) = \det{(A)}^2$$

$$\det(A^{-1}) = \frac{1}{\det{(A')}}$$

Expand the exponential of exp(A) as a Maclaurin series of square matrices:

$$e^{A} = I + A + \frac{1}{2}A^2 + \cdots = \sum_{m=0}^{\infty} \frac{1}{m!}A^m$$

$$\det(e^{A}) = \sum_{m=0}^{\infty} \frac{1}{m!}\det{(A^m)}$$

The trace equation is given as shown below:

$$tr(A) = \sum_{m=1}^{n} A_{mm}$$

$$e^{tr(A)} = e^{\sum_{m=1}^{n} A_{mm}} = e^{A_{11}}e^{A_{22}} \cdots e^{A_{nn}}$$

Now suppose:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$e^{tr(A)} = e^a e^d$$

$$\det(e^A) = \sum_{m=0}^{\infty} \frac{1}{m!} \det(A^m) = \sum_{m=0}^{\infty} \frac{1}{m!} \det(A)^m = n + ad - bc + \frac{1}{2}(ad - bc)^2 + \cdots$$

I must rely on the reader to finish solving this problem.

4. Find two matrices A and B that satisfy the following equations:

$$A^2 = 0, A\hat{A} + \hat{A}A = I, B = \hat{A}A$$

$$(A_{kl})^2 = \sum_{m=1}^{n} A_{km} A_{ml}$$

$$A\hat{A} + \hat{A}A = \sum_{m=1}^{n} A_{km} A_{ml}^* + \sum_{m=1}^{n} A_{km}^* A_{ml}$$

Where 0 is the null matrix and I is the unit (identity) matrix. Obtain explicit expressions for A and B in a representation in which B is diagonal, assuming it is nondegenerate. Can A be diagonalized in any representation?

Suppose A is:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{bmatrix} = \begin{bmatrix} a^2 + bc & 2ab \\ 2ac & a^2 + cb \end{bmatrix}$$

$$A\hat{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix} = \begin{bmatrix} aa^* + bc^* & ab^* + bd^* \\ ca^* + dc^* & cb^* + dd^* \end{bmatrix}$$

$$\hat{A}A = \begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} aa^* + b^*c & a^*b + b^*d \\ ac^* + d^*c & c^*b + d^*d \end{bmatrix}$$

$$A\hat{A} + \hat{A}A = \begin{bmatrix} 2aa^* + bc^* + b^*c & ab^* + a^*b + b^*d + bd^* \\ ca^* + \ ac^* + dc^* + d^*c & cb^* + c^*b + 2dd^* \end{bmatrix}$$

$$2aa^* + bc^* + b^*c = 1$$

$$ab^* + a^*b + b^*d + bd^* = 0$$

$$ca^* + ac^* + dc^* + d^*c = 0$$

$$cb^* + c^*b + 2dd^* = 1$$

$$2aa^* - 2dd^* = 0$$

$$aa^* - dd^* = 0$$

$$a = d$$

$$b = c = 0$$

$$2aa^* = 1$$

$$aa^* = \frac{1}{2}$$

$$a = x + iy, a^* = x - iy, aa^* = x^2 + y^2 = \frac{1}{2}, x = y = \frac{1}{\sqrt{2}}$$

$$a = d = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, b = c = 0$$

In the general case:

$$A\hat{A} + \hat{A}A = 2A_{kl}A_{kl}^*\delta_{kl} = \delta_{kl}$$

$$A_{kk} = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\delta_{kl}$$

$$\delta_{kl} = 1 \forall k = l$$

$$\delta_{kl} = 0 \forall k \neq l$$

$$B_{kl} = \left(\hat{A}A\right)_{kl} = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\delta_{kl}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\delta_{kl} = \left(\frac{1}{2} + \frac{1}{2}\right)\delta_{kl} = \delta_{kl}$$

$$B = I$$

5.Find three matrices A, B, and C that satisfy the following equations:

$$A^2 = B^2 = C^2 = I, AB + BA = BC + CB = CA + AC = 0$$

Where 0 is the null matrix. Obtain explicit expressions for all three matrices in a representation in which A is diagonal , assuming it is nondegenerate.

$$A^2 = \sum_{m=1}^{n} A_{km} A_{ml} = \delta_{kl}$$

$$B^2 = \sum_{m=1}^{n} B_{km} B_{ml} = \delta_{kl}$$

$$C^2 = \sum_{m=1}^{n} C_{km} C_{ml} = \delta_{kl}$$

$$AB + BA = \sum_{m=1}^{n} A_{km} B_{ml} + \sum_{m=1}^{n} B_{km} A_{ml} = 0$$

$$BC + CB = \sum_{m=1}^{n} B_{km} C_{ml} + \sum_{m=1}^{n} C_{km} B_{ml} = 0$$

$$CA + AC = \sum_{m=1}^{n} C_{km} A_{ml} + \sum_{m=1}^{n} A_{km} C_{ml} = 0$$

$$A_{km} B_{ml} + B_{km} A_{ml} = 0$$

$$B_{km} C_{ml} + C_{km} B_{ml} = 0$$

$$C_{km} A_{ml} + A_{km} C_{ml} = 0$$

$$SAS^{-1} = A'$$

$$SA_{kl}S^{-1} = A'_{kl}\delta_{kl}$$
$$A_{km}B_{ml} = -B_{km}A_{ml}$$

$$\sum_{m=1}^{n} S_{km}A_{ml} = A'_k S_{kl}$$

Suppose the following:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}, C = \begin{bmatrix} r & s \\ t & u \end{bmatrix}$$

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

$$BA = \begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea + fc & eb + fd \\ ga + hc & gb + hd \end{bmatrix}$$

$$BC = \begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} er + ft & es + fu \\ gr + ht & gs + hu \end{bmatrix}$$

$$CB = \begin{bmatrix} r & s \\ t & u \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} re + sg & rf + sh \\ te + ug & tf + uh \end{bmatrix}$$

$$CA = \begin{bmatrix} r & s \\ t & u \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ra + sc & rb + sd \\ ta + uc & tb + ud \end{bmatrix}$$

$$AC = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} ar + bt & as + bu \\ cr + dt & cs + du \end{bmatrix}$$

▷ How to Diagonalize a Matrix ✅ (with practice problems) (algebrapracticeproblems.com)

We use an example from the previous website:

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - 2 = 4 - 5\lambda + \lambda^2 = (\lambda-4)(\lambda-1)$$

The roots of the eigenvalue equation are:

$$\lambda_1 = 4, \lambda_2 = 1$$

$$(A - \lambda_1 I)v = 0$$

$$\begin{bmatrix} 2-4 & 2 \\ 1 & 3-4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2x+2y \\ 1-y \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Likewise:

$$\begin{bmatrix} 2-1 & 2 \\ 1 & 3-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ x+2y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$S = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$S^{-1} = \frac{1}{-2-1}\begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}+\frac{1}{3} & 0 \\ 0 & \frac{2}{3}+\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

I leave it up to reader to complete this problem.