Solution of a Second Order Self-Adjoint Ordinary Differential Equation (ODE) via Three Methods

The ODE is of the form:

$$-\frac{d^2y}{dx^2}+r(x)y(x)=f(x)\forall a\leq x\leq b, r(x)\geq 0\forall a\leq x\leq b$$

We solve the equation:

$$-\frac{d^2y}{dx^2}+\sin(x)\,y(x)=\cos(x)\,,0\leq x\leq 0.1\pi, y(0)=1, y'(0)=1, y(0.1\pi)=1, y'(0.1\pi)=1$$

Using an infinite power series that satisfies the equation:

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

The first derivative is:

$$\frac{dy}{dx}=\sum_{n=0}^{\infty}na_nx^{n-1}$$

The second derivative maybe found as follows:

$$\frac{d^2y}{dx^2}=\sum_{n=0}^{\infty}n(n-1)a_nx^{n-2}$$

The boundary conditions are:

$$y(0)=a_0=1$$

$$y'(0)=a_1=1$$

$$y(0.1\pi)=1$$

$$y'(0.1\pi)=1$$

Filling in the ODE with the series approximations yields:

$$\sum_{n=0}^{\infty}[-n(n-1)a_nx^{n-2}+\sin(x)\,a_nx^n]=\cos(x)$$

Now we need to expand the trigonometric functions using infinite power series (Maclaurin series):

$$r(\mathrm{x})=\sin(x)\,,r(0)=0$$

$$r'(x) = \cos(x)\,, r'(0) = 1$$

$$r''(x) = -\sin(x)\,, r''(0) = 0$$

$$r'''(x) = -\cos(x)\,, r'''(0) = -1$$

$$r'^{v}(x) = \sin(x), r'^{v}(0) = 0$$

$$r^{v}(x) = \cos(x)\,, r^{v}(0) = 1$$

$$r^{vi}(x) = -\sin(x)\,, r^{vi}(0) = 0$$

$$r^{vii}(x) = -\cos(x), r^{vii}(0) = 1$$

$$r^{viii}(x) = \sin(x)\,, r^{viii}(0) = 0$$

$$r(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$f(x) = \cos(x)\,, f(0) = 1$$

$$f'(\mathrm{x}) = -\sin(x)\,, f'(0) = 0$$

$$f''(\mathrm{x}) = -\cos(x)\,, f''(0) = -1$$

$$f'''(\mathrm{x}) = \sin(x), f'''(0) = 0$$

$$f^{iv}(\mathrm{x}) = \cos(x), f^{iv}(0) = 1$$

$$f^{v}(\mathrm{x}) = -\sin(x)\,, f^{v}(0) = 0$$

$$f^{vi}(\mathrm{x}) = -\cos(x), f^{vi}(0) = -1$$

$$f^{vii}(\mathrm{x}) = \sin(x), f^{vii}(0) = 0$$

$$f^{viii}(\mathrm{x}) = \cos(x)\,, f^{viii}(0) = 1$$

$$f(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

[Calculus II - Series - The Basics (lamar.edu)]

$$y(x)\sin(x) = \sum_{n=0}^{\infty} a_n x^n \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$= (a_0 + a_1 x + a_2 x^2 + a_2 x^3 + \cdots)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\right)$$

$$= a_0\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\right) + a_1 x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\right)$$

$$+ a_2 x^2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\right) + a_3 x^3\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\right)$$

$$+ a_4 x^4\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\right) + a_5 x^5\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\right)$$

$$+ a_6 x^6\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\right) + a_7 x^7\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\right)$$

$$+ a_8 x^8\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\right) + \cdots$$

$$= a_1 x^2 - a_1 \frac{x^4}{3!} + a_3 x^4 + a_1 \frac{x^6}{5!} - a_3 \frac{x^6}{3!} - a_1 \frac{x^8}{7!} + a_3 \frac{x^8}{5!} - a_5 \frac{x^8}{3!} + a_7 x^8$$

$$- a_3 \frac{x^{10}}{7!} + a_5 \frac{x^{10}}{5!} - a_7 \frac{x^{10}}{3!} + a_9 x^{10} - \cdots =$$

$$a_0 = 1$$

$$a_1 = 1$$

$$-2a_2 = 1, a_2 = -\frac{1}{2}$$

$$-3 \cdot 2a_3 x + a_0 x = 0, -6a_3 + 1 = 0, a_3 = \frac{1}{6}$$

$$-4 \cdot 3a_4 x^2 + a_1 x^2 = -\frac{1}{2!}x^2, -12a_4 + 1 = -\frac{1}{2}, -12a_4 = -\frac{2}{2} - \frac{1}{2} = -\frac{3}{2}, a_4 = \frac{3}{24} = \frac{1}{8}$$

$$-5 \cdot 4a_5 x^3 + a_2 x^3 - \frac{1}{3!}a_0 = 0, -20a_5 - \frac{1}{2} - \frac{1}{6} = 0, a_5 = -\frac{1}{20} \cdot \left(\frac{3}{6} + \frac{1}{6}\right) = -\frac{1}{20} \cdot \frac{4}{6} = -\frac{1}{30}$$

$$-6 \cdot 5a_6 x^4 + a_3 x^4 - \frac{1}{3!}a_1 x^4 = \frac{1}{4!}x^4, -30a_6 + \frac{1}{6} - \frac{1}{6} = \frac{1}{24}, a_6 = -\frac{1}{30} \cdot \frac{1}{24} = -\frac{1}{720}$$

$$-7 \cdot 6a_7 x^5 + a_4 x^5 - \frac{1}{3!}a_2 x^5 + a_0 \frac{1}{5!} = 0, -42a_7 + \frac{1}{8} + \frac{1}{5!} = 0, a_7 = -\frac{1}{8} - \frac{1}{120}, a_7$$
$$= -\frac{1}{8} - \frac{1}{120}$$

$$-8\cdot 7a_8x^6 + a_5x^6 - a_3\frac{x^6}{3!} + \frac{1}{5!}a_1x^6 = -\frac{x^6}{6!}, -56a_8 - \frac{1}{30} - \frac{1}{6}\cdot\frac{1}{6} + \frac{1}{120} = -\frac{1}{720}, -56a_8$$
$$= -\frac{1}{720} + \frac{1}{30} + \frac{1}{36} - \frac{1}{120}, a_8 = -\frac{1}{56}\left(-\frac{1}{720} + \frac{1}{30} + \frac{1}{36} - \frac{1}{120}\right)$$

$$-9\cdot 8a_9x^7 + a_6x^7 - a_4\frac{x^7}{3!} + a_0x^7 = 0, -72a_9 = \frac{7}{720} - \frac{1}{8}\cdot\frac{1}{6} - 1, a_9$$
$$= -\left(\frac{7}{720} - \frac{1}{8}\cdot\frac{1}{6} - 1\right)\frac{1}{72}$$

$$-10\cdot 9a_{10}x^8 - a_1\frac{x^8}{7!} - a_5\frac{x^8}{3!} + a_7x^8 = \frac{x^8}{8!}, -90a_{10} - \frac{1}{7!} + \frac{1}{30\cdot 6} - \frac{1}{8} - \frac{1}{120} = \frac{1}{8!}, a_{10}$$
$$= -\frac{1}{90}\cdot\left(\frac{1}{5040} - \frac{1}{180} + \frac{1}{8} + \frac{1}{120}\right)$$

In our computer solution to this problem, we use a function from *A Numerical Library in C for Scientists and Engineers* © 1994 by H. T. Lau, PhD. Lau's function utilizes the Galerkin method with piecewise continuous polynomials. Below is the Cauchy product of two infinite power series.

Cauchy product - Wikipedia

$$\sum_{i=0}^{\infty} a_i x^i \sum_{j=0}^{\infty} b_j x^j = \sum_{k=0}^{\infty} c_k x^k$$

$$c_k = \sum_{l=0}^{k} a_l b_{k-l}$$

$$\sin(x)\, y(x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} x^{2i+1} \sum_{j=0}^{\infty} a_j x^j = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} x^{2i+1}\, \delta_{i,2i+1} \sum_{j=0}^{\infty} a_j x^j$$

$$c_k = \sum_{l=0}^{k} \frac{(-1)^l}{(2l+1)!} \delta_{l,2l+1} a_{k-l}$$

$$\sum_{k=0}^{\infty} [-k(k-1)a_k x^{k-2} + c_k x^k] = \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l)!} \delta_{k,2l} x^{2l}$$

$$-k(k-1)a_k x^k + c_k x^{k+2} = \frac{(-1)^l}{(2l)!} \delta_{k,2l} x^{2l+2}$$

$$a_k = -\frac{1}{k(k-1)}\left[\sum_{l=0}^{k} \frac{(-1)^l}{(2l)!} \delta_{k,2l}\, c_{k-2} - c_k\right]$$

Here are the results discovered by using sixteen terms of the Cauchy product formula, eleven terms of the infinite power series, and then the femlag function. The first, third, and fifth columns are x-values. The second, fourth, and sixth columns are the y-values:

Cauchy x	Cauchy y	Series x	Series y	femlag x	femlag y
0.000000e+00	1.000000e+00	0.000000e+00	1.000000e+00	0.000000e+00	1.074421e+00
1.570796e-02	1.015708e+00	1.570796e-02	9.998773e-01	1.570796e-02	1.088838e+00
3.141593e-02	1.031416e+00	3.141593e-02	9.995118e-01	3.141593e-02	1.103012e+00
4.712389e-02	1.047124e+00	4.712389e-02	9.989077e-01	4.712389e-02	1.116947e+00
6.283185e-02	1.062832e+00	6.283185e-02	9.980693e-01	6.283185e-02	1.130650e+00
7.853982e-02	1.078540e+00	7.853982e-02	9.970011e-01	7.853982e-02	1.144123e+00
9.424778e-02	1.094248e+00	9.424778e-02	9.957078e-01	9.424778e-02	1.157373e+00
1.099557e-01	1.109956e+00	1.099557e-01	9.941941e-01	1.099557e-01	1.170404e+00
1.256637e-01	1.125664e+00	1.256637e-01	9.924651e-01	1.256637e-01	1.183221e+00
1.413717e-01	1.141372e+00	1.413717e-01	9.905258e-01	1.413717e-01	1.195831e+00
1.570796e-01	1.157080e+00	1.570796e-01	9.883815e-01	1.570796e-01	1.208237e+00
1.727876e-01	1.172788e+00	1.727876e-01	9.860376e-01	1.727876e-01	1.220447e+00
1.884956e-01	1.188496e+00	1.884956e-01	9.834996e-01	1.884956e-01	1.232465e+00
2.042035e-01	1.204204e+00	2.042035e-01	9.807731e-01	2.042035e-01	1.244298e+00
2.199115e-01	1.219911e+00	2.199115e-01	9.778637e-01	2.199115e-01	1.255951e+00
2.356194e-01	1.235619e+00	2.356194e-01	9.747773e-01	2.356194e-01	1.267431e+00
2.513274e-01	1.251327e+00	2.513274e-01	9.715197e-01	2.513274e-01	1.278745e+00
2.670354e-01	1.267035e+00	2.670354e-01	9.680967e-01	2.670354e-01	1.289898e+00
2.827433e-01	1.282743e+00	2.827433e-01	9.645142e-01	2.827433e-01	1.300897e+00
2.984513e-01	1.298451e+00	2.984513e-01	9.607780e-01	2.984513e-01	1.311748e+00
3.141593e-01	1.000000e+00	3.141593e-01	1.000000e+00	3.141593e-01	1.322459e+00

```
Minimum Absolute Cauchy Error =  1.329683e-02
Minimum Absolute Series Error =  3.224590e-01
Average Absolute Cauchy Error =  6.564407e-02
Average Absolute Series Error =  2.287446e-01
Maximum Absolute Cauchy Error =  3.509701e-01
Maximum Absolute Series Error =  3.509701e-01

Galerkin Order = 6      RMSE Cauchy =  8.950036e-02
Galerkin Order = 6      RMSE Series =  2.396865e-01
```

Next, we solve the initial value problem using three methods:

1. Series
2. Cauchy product series
3. 5th Order Runge-Kutta

The initial values are:

$$y(0) = a_0 = 1$$

$$y'(0) = a_1 = 1$$

The function is:

$$f(x, y) = y(x)\sin x - \cos x$$

The equation solved using the initial conditions is as follows:

$$\frac{d^2y}{dx^2} = f(x, y)$$

$$a_k = \frac{1}{k(k-1)}\left[c_k - \sum_{l=0}^{k} \frac{(-1)^l}{(2l)!}\,\delta_{k,2l}\,c_{k-2}\right]$$

We obtain the following results for the initial values problem:

Cauchy x	Cauchy y	Series x	Series y	femlag x	femlag y
0.000000e+00	1.000000e+00	0.000000e+00	1.000000e+00	0.000000e+00	1.000000e+00
1.570796e-02	1.015708e+00	1.570796e-02	1.015585e+00	1.570796e-02	1.015585e+00
3.141593e-02	1.031416e+00	3.141593e-02	1.030928e+00	3.141593e-02	1.030928e+00
4.712389e-02	1.047124e+00	4.712389e-02	1.046032e+00	4.712389e-02	1.046032e+00
6.283185e-02	1.062832e+00	6.283185e-02	1.060901e+00	6.283185e-02	1.060901e+00
7.853982e-02	1.078540e+00	7.853982e-02	1.075541e+00	7.853982e-02	1.075541e+00
9.424778e-02	1.094248e+00	9.424778e-02	1.089956e+00	9.424778e-02	1.089956e+00
1.099557e-01	1.109956e+00	1.099557e-01	1.104150e+00	1.099557e-01	1.104150e+00
1.256637e-01	1.125664e+00	1.256637e-01	1.118129e+00	1.256637e-01	1.118129e+00
1.413717e-01	1.141372e+00	1.413717e-01	1.131897e+00	1.413717e-01	1.131898e+00
1.570796e-01	1.157080e+00	1.570796e-01	1.145461e+00	1.570796e-01	1.145461e+00
1.727876e-01	1.172788e+00	1.727876e-01	1.158825e+00	1.727876e-01	1.158826e+00
1.884956e-01	1.188496e+00	1.884956e-01	1.171995e+00	1.884956e-01	1.171996e+00
2.042035e-01	1.204204e+00	2.042035e-01	1.184977e+00	2.042035e-01	1.184979e+00
2.199115e-01	1.219911e+00	2.199115e-01	1.197775e+00	2.199115e-01	1.197779e+00
2.356194e-01	1.235619e+00	2.356194e-01	1.210397e+00	2.356194e-01	1.210402e+00
2.513274e-01	1.251327e+00	2.513274e-01	1.222847e+00	2.513274e-01	1.222856e+00
2.670354e-01	1.267035e+00	2.670354e-01	1.235132e+00	2.670354e-01	1.235145e+00
2.827433e-01	1.282743e+00	2.827433e-01	1.247257e+00	2.827433e-01	1.247277e+00
2.984513e-01	1.298451e+00	2.984513e-01	1.259229e+00	2.984513e-01	1.259258e+00
3.141593e-01	1.314159e+00	3.141593e-01	1.271053e+00	3.141593e-01	1.271095e+00

Minimum Absolute Cauchy Error = 0.000000e+00
Minimum Absolute Series Error = 0.000000e+00
Average Absolute Cauchy Error = 1.602394e-02
Average Absolute Series Error = 6.282434e-06
Maximum Absolute Series Error = 4.141360e-05
Maximum Absolute Series Error = 4.141360e-05

RMSE Series = 7.982836e-04
RMSE Cauchy = 3.237282e-02