

Blog Entry Friday, June 7, 2022, A Nice Integral © James Pate Williams, Jr. All Applicable Rights Reserved

We evaluate the definite improper integral:

$$I = \int_0^{\infty} e^{-x^2} dx$$

We square the integral:

$$I^2 = \frac{1}{2} \int_0^{\infty} \int_0^{\infty} e^{-x^2-y^2} dx dy$$

Noting that the following holds:

$$r^2 = x^2 + y^2$$

Next, we transform from Cartesian coordinates to polar coordinates:

$$x = r \cos \theta$$

$$y = r \sin \theta$$

The line element is defined by the equation:

$$ds^2 = dx^2 + dy^2$$

Differentiation yields the following equations:

$$dx = \cos \theta dr - r \sin \theta d\theta$$

$$dy = \sin \theta dr + r \cos \theta d\theta$$

$$dx^2 = (\cos \theta)^2 dr^2 - 2r \cos \theta \sin \theta dr d\theta + r^2 (\sin \theta)^2 d\theta^2$$

$$dy^2 = (\sin \theta)^2 dr^2 + 2r \cos \theta \sin \theta dr d\theta + r^2 (\cos \theta)^2 d\theta^2$$

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$$

$$dx dy = r dr d\theta$$

$$I^2 = \frac{1}{2} \int_0^{\infty} \int_0^{\pi} e^{-r^2} r dr d\theta = \frac{\pi}{4}$$

$$I = \frac{\sqrt{\pi}}{2} \approx 0.8862269254$$

I find using adaptive extended Simpson's Rule and extended Simpson's Rule the following approximations:

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f(x) = e^(-x*x) a = 0, b = 100
Exact Value = 0.886226925453
Adaptive Ext Simpson's Rule 16 = 0.702223048723
Error 16 = 0.184003876729
Adaptive Ext Simpson's Rule 32 = 0.875857083838
Error 32 = 0.010369841614
Adaptive Ext Simpson's Rule 48 = 0.886160675431
Error 48 = 0.000066250022
Adaptive Ext Simpson's Rule 64 = 0.886226869380
Error 64 = 0.000000056073
Adaptive Ext Simpson's Rule 80 = 0.886226925446
Error 80 = 0.000000000006
Adaptive Ext Simpson's Rule 96 = 0.886226925446
Error 96 = 0.000000000006
Extended Simpson's Rule 16 = 2.083333333333
Error 16 = 1.197106407881
Extended Simpson's Rule 32 = 1.041905795370
Error 32 = 0.155678869917
Extended Simpson's Rule 48 = 0.730647005206
Error 48 = 0.155579920247
Extended Simpson's Rule 64 = 0.702223048723
Error 64 = 0.184003876729
Extended Simpson's Rule 80 = 0.767628992292
Error 80 = 0.118597933161
Extended Simpson's Rule 96 = 0.767628992292
Error 96 = 0.118597933161
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