Blog Entry © Tuesday, October 8, 2024, by James Pate Williams, Jr.

Calculate a 3 by 3 System of Coupled First Order Linear Ordinary Differential Equations Zeros of the Characteristic Polynomial (Compute the Eigenvalues)

$$\dot{\vec{x}} = A\vec{x}$$

$$det|A - \lambda I| = 0$$

For a coupled linear 3 by 3 system yields:

$det$|A-λI|= $\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} \begin{matrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \\ a_{31} & a_{32} \end{matrix} = (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda) + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{21}(a_{33}-\lambda) - (a_{11}-\lambda)a_{23}a_{32} - a_{13}(a_{22}-\lambda)a_{31} = [a_{11}a_{22} - (a_{22}+a_{11})\lambda + \lambda^2](a_{33}-\lambda) + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{21}\lambda - a_{11}a_{23}a_{32} + a_{23}a_{32}\lambda - a_{13}a_{22}a_{31} + a_{13}a_{31}\lambda = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{22}\lambda - (a_{22}+a_{11})a_{33}\lambda + a_{12}a_{21}\lambda + a_{23}a_{32}\lambda + a_{13}a_{31}\lambda + (a_{22}+a_{11})\lambda^2 + a_{33}\lambda^2 - \lambda^3 = 0$

The determinant is:

$$\det = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31}$$

The characteristic equation is a cubic equation of the form:

$$az^3 + bz^2 + cz + d = 0$$

$$a = -1, b = a_{22} + a_{11} + a_{33}, c = -a_{11}a_{22} - (a_{22}+a_{11})a_{33} + a_{12}a_{21} + a_{23}a_{32} + a_{13}a_{31}, d = det$$

We use an implementation of Mueller's Method which is found in *A Numerical Library in C for Scientists and Engineers* © 1994 by H. T. Lau, PhD. The algorithm finds the complex eigenvalues.

See

[{{1,2,3},{3,2,1},{2,1,3}} - Wolfram|Alpha (wolframalpha.com)]

$$|A - \lambda I| = 0$$

We reproduce the Wolfram calculation by hand then by machine:

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 2-\lambda & 1 \\ 2 & 1 & 3-\lambda \end{vmatrix} \begin{matrix} 1-\lambda & 2 \\ 3 & 2-\lambda \\ 2 & 1 \end{matrix} = 0$$

Expanding the previous determinant yields the characteristic polynomial:

$$\begin{aligned}|A-\lambda I| &= (1-\lambda)(2-\lambda)(3-\lambda)+4+9-6(3-\lambda)-(1-\lambda)-6(2-\lambda)\\ &= (2-2\lambda-\lambda+\lambda^2)(3-\lambda)+13-18+6\lambda-1+\lambda-12+6\lambda\\ &= 6-6\lambda-3\lambda+3\lambda^2-\lambda+2\lambda^2+\lambda^2-\lambda^3+6\lambda-18+13-13+6\lambda\\ &= -10\lambda+12\lambda+6\lambda^2-\lambda^3+6-18=-\lambda^3+6\lambda^2+2\lambda-12\end{aligned}$$

$$|A| = 6+4+9-18-1-12=-12$$

The characteristic equation checked for accuracy as follows:

$$b = a_{22}+a_{11}+a_{33}=2+1+3=6$$

$$\begin{aligned}c &= -a_{11}a_{22}-(a_{22}+a_{11})a_{33}+a_{12}a_{21}+a_{23}a_{32}+a_{13}a_{31}=-2-3(2+1)+6+1+6\\ &= -2-6-3+13=-8+10=2\end{aligned}$$

$$d=-1$$

Now we need to compute the solutions to the system of first order linear coupled ordinary differential equations:

$$(A-\lambda I)v=0$$

$$Av=\lambda v$$

$$v_1=\frac{1}{det}\left|\begin{array}{ccc|cc} x_1 & a_{12} & a_{13} & x_1 & a_{12}\\ 0 & a_{22}-\lambda & a_{23} & 0 & a_{22}-\lambda\\ 0 & a_{32} & a_{33}-\lambda & 0 & a_{32}\end{array}\right. = \frac{1}{det}[(a_{22}-\lambda)(a_{33}-\lambda)-a_{23}a_{32}]x_1$$

$$v_2=\frac{1}{det}\left|\begin{array}{ccc|cc} a_{11}-\lambda & 0 & a_{13} & a_{11} & 0\\ a_{21} & x_2 & a_{23} & a_{21} & x_2\\ a_{31} & 0 & a_{33}-\lambda & a_{31} & 0\end{array}\right. = \frac{1}{det}[(a_{11}-\lambda)(a_{33}-\lambda)-a_{13}a_{31}]x_2$$

$$v_3=\frac{1}{det}\left|\begin{array}{ccc|cc} a_{11} & a_{12} & 0 & a_{11} & a_{12}\\ a_{21} & a_{22}-\lambda & 0 & a_{21} & a_{22}-\lambda\\ a_{31} & a_{32} & x_3 & a_{11} & a_{32}\end{array}\right. = \frac{1}{det}[a_{11}(a_{22}-\lambda)-a_{21}a_{12}]x_3$$

$$v_1=-\frac{1}{12}\cdot[(2-\lambda)(3-\lambda)-1]$$

$$v_2=-\frac{1}{12}\cdot[(1-\lambda)(3-\lambda)-9]$$

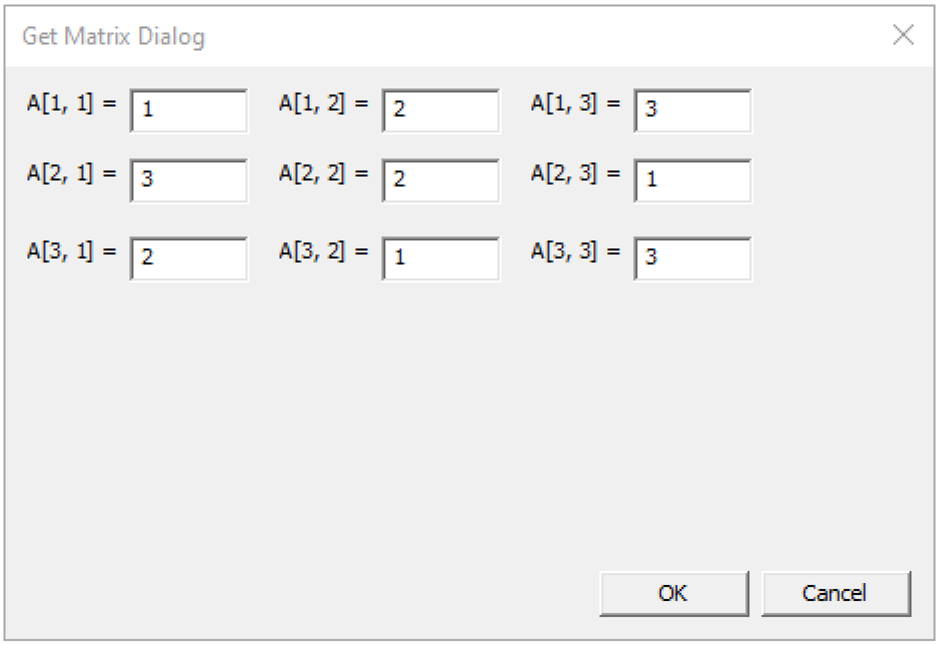
$$v_3=-\frac{1}{12}\cdot[(2-\lambda)-3\cdot 2]$$

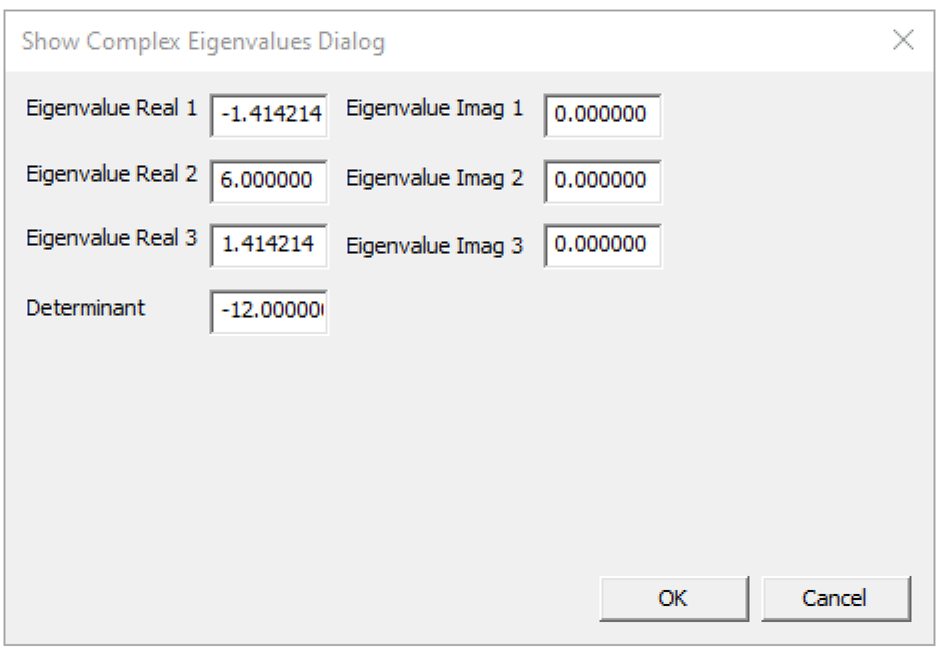
Using numerical analysis, e.g., executing an application which contains a C function in *A Numerical Library in C for Scientists and Engineers* © 1994 by H. T. Lau, PhD, that finds the complex and/or real roots of a polynomial via Mueller's Method, we obtain the following eigenvalues:

$$\lambda = -\sqrt{2}, \sqrt{2}, 6$$

The approximate eigenvalues are to five decimal places:

$$\lambda = -1.41421, 1.41421, 6.00000$$



```
original matrix:

+1.00000e+00   +2.00000e+00   +3.00000e+00
+3.00000e+00   +2.00000e+00   +1.00000e+00
+2.00000e+00   +1.00000e+00   +3.00000e+00

real transformed matrix:

-1.41421e+00   +2.32588e-01   -1.12959e+00
+6.60555e+00   +6.00000e+00   +8.18489e-01
+2.00000e+00   -8.46154e-01   +1.41421e+00

imag transformed matrix:

+0.00000e+00   +0.00000e+00   +0.00000e+00
+0.00000e+00   +0.00000e+00   +0.00000e+00
+0.00000e+00   +0.00000e+00   +0.00000e+00

eigenvalues:

-1.41421e+00   +0.00000e+00
+6.00000e+00   +0.00000e+00
+1.41421e+00   +0.00000e+00

determinant = -12

eigenvectors:

+1.00000e+00   0
-7.98990e-01   0
-2.72078e-01   0

+1.00000e+00   0
+1.00000e+00   0
+1.00000e+00   0

+2.57739e-02   0
+1.00000e+00   0
-6.63108e-01   0
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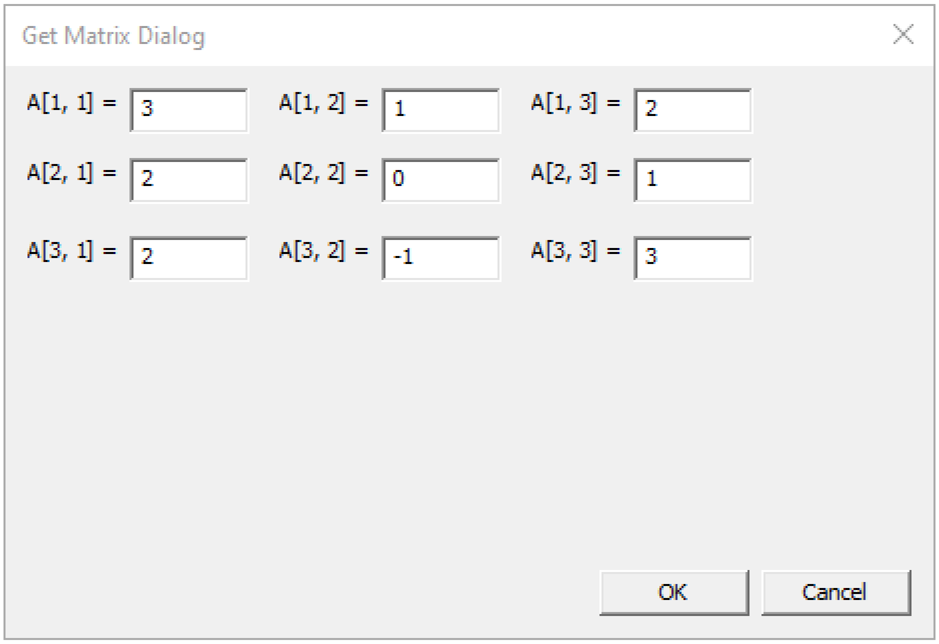
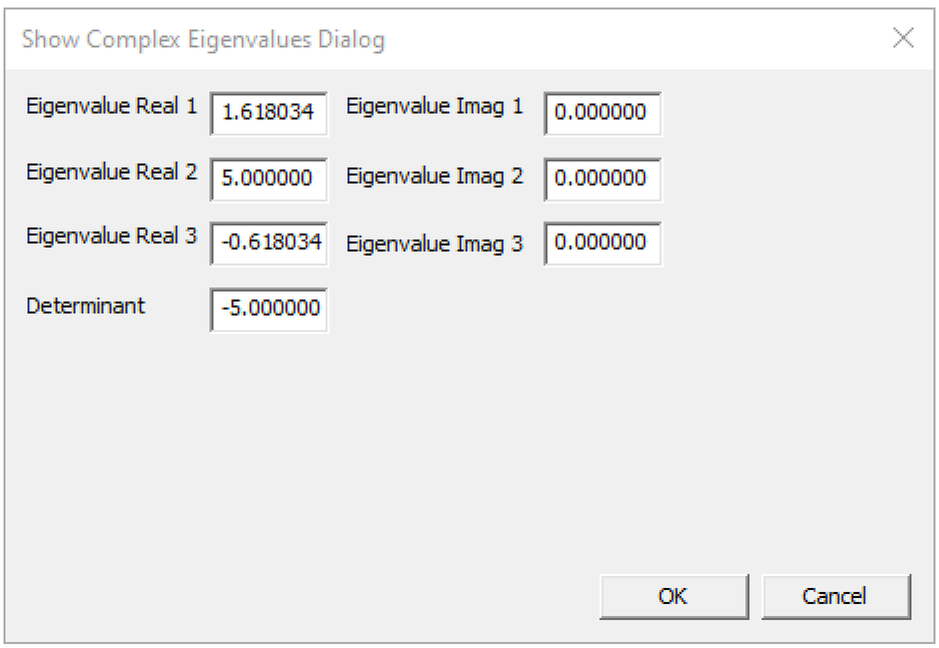
Another online example calculator:

System of ODEs Calculator (symbolab.com)


Get Matrix Dialog
A[1, 1] = 3
A[1, 2] = 1
A[1, 3] = 2
A[2, 1] = 2
A[2, 2] = 0
A[2, 3] = 1
A[3, 1] = 2
A[3, 2] = -1
A[3, 3] = 3
OK
Cancel


Show Complex Eigenvalues Dialog
Eigenvalue Real 1 1.618034
Eigenvalue Imag 1 0.000000
Eigenvalue Real 2 5.000000
Eigenvalue Imag 2 0.000000
Eigenvalue Real 3 -0.618034
Eigenvalue Imag 3 0.000000
Determinant -5.000000
OK
Cancel

```
original matrix:

+3.00000e+00   +1.00000e+00   +2.00000e+00
+2.00000e+00   +0.00000e+00   +1.00000e+00
+2.00000e+00   -1.00000e+00   +3.00000e+00

real transformed matrix:

-6.18034e-01   +1.20659e+00   +1.26304e+00
+4.82843e+00   +5.00000e+00   +1.39602e+00
+2.00000e+00   -5.00000e-01   +1.61803e+00

imag transformed matrix:

+0.00000e+00   +0.00000e+00   +0.00000e+00
+0.00000e+00   +0.00000e+00   +0.00000e+00
+0.00000e+00   +0.00000e+00   +0.00000e+00

eigenvalues:

-6.18034e-01   +0.00000e+00
+5.00000e+00   +0.00000e+00
+1.61803e+00   +0.00000e+00

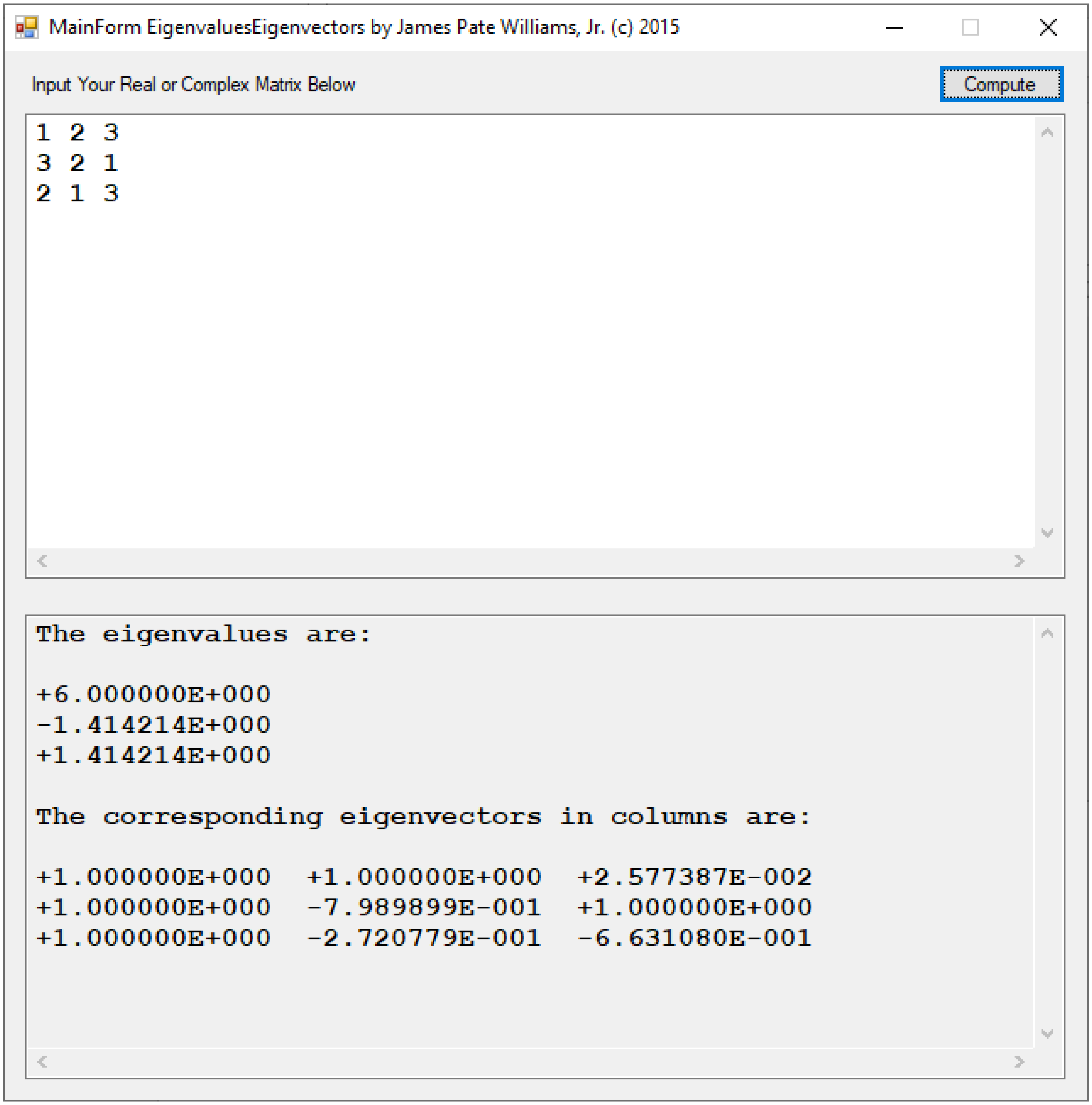
determinant = -5

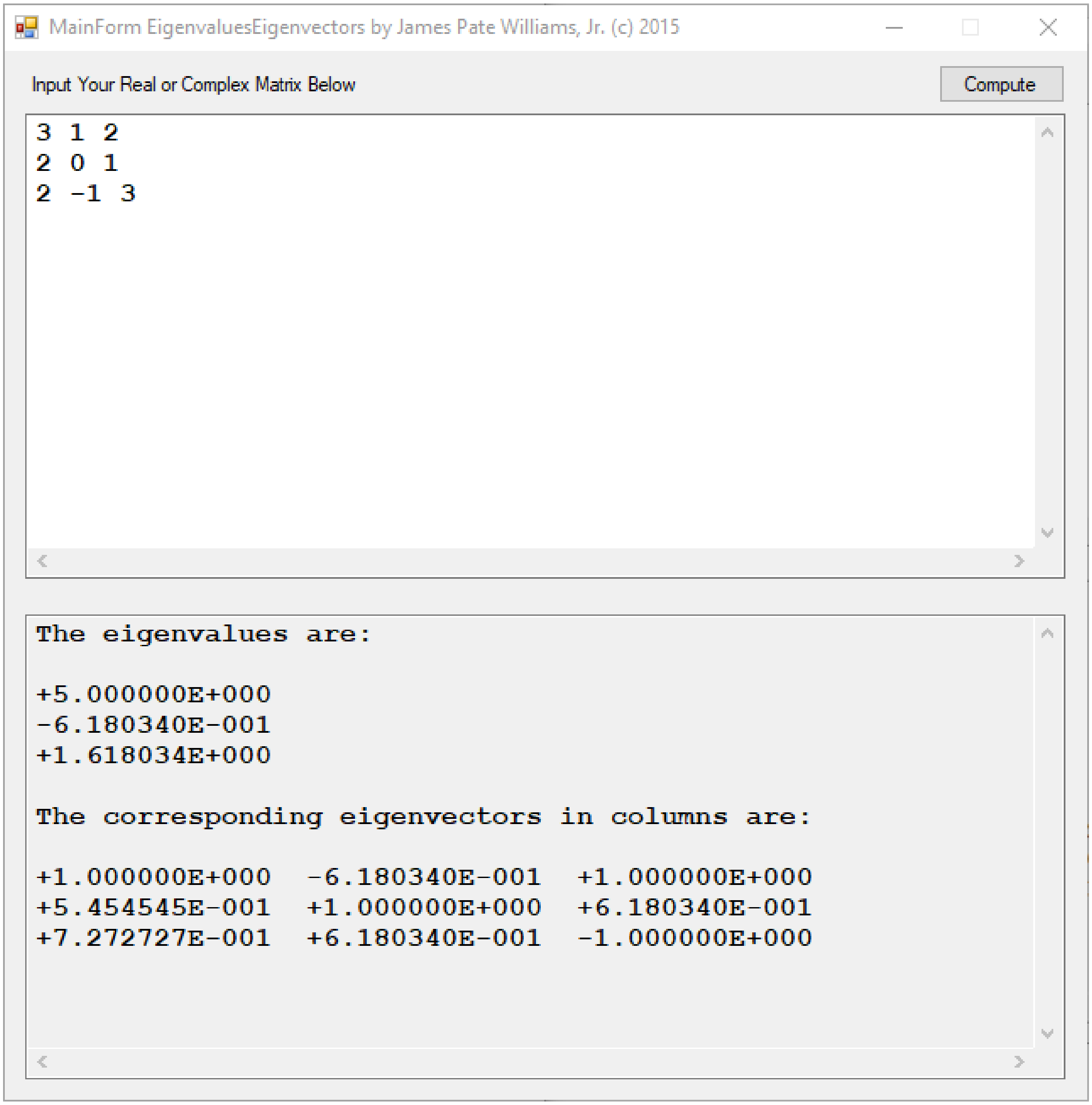
eigenvectors:

-6.18034e-01   0
+1.00000e+00   0
+6.18034e-01   0

+1.00000e+00   0
+5.45455e-01   0
+7.27273e-01   0

-1.00000e+00   0
-6.18034e-01   0
+1.00000e+00   0
```


MainForm EigenvaluesEigenvectors by James Pate Williams, Jr. (c) 2015
Input Your Real or Complex Matrix Below
Compute
1 2 3
3 2 1
2 1 3
The eigenvalues are:
+6.000000E+000
-1.414214E+000
+1.414214E+000
The corresponding eigenvectors in columns are:
+1.000000E+000 +1.000000E+000 +2.577387E-002
+1.000000E+000 -7.989899E-001 +1.000000E+000
+1.000000E+000 -2.720779E-001 -6.631080E-001


MainForm EigenvaluesEigenvectors by James Pate Williams, Jr. (c) 2015
Input Your Real or Complex Matrix Below
Compute
3 1 2
2 0 1
2 -1 3
The eigenvalues are:
+5.000000E+000
-6.180340E-001
+1.618034E+000
The corresponding eigenvectors in columns are:
+1.000000E+000 -6.180340E-001 +1.000000E+000
+5.454545E-001 +1.000000E+000 +6.180340E-001
+7.272727E-001 +6.180340E-001 -1.000000E+000