Test Particle and Photon Traversal about a Schwarzschild Singularity

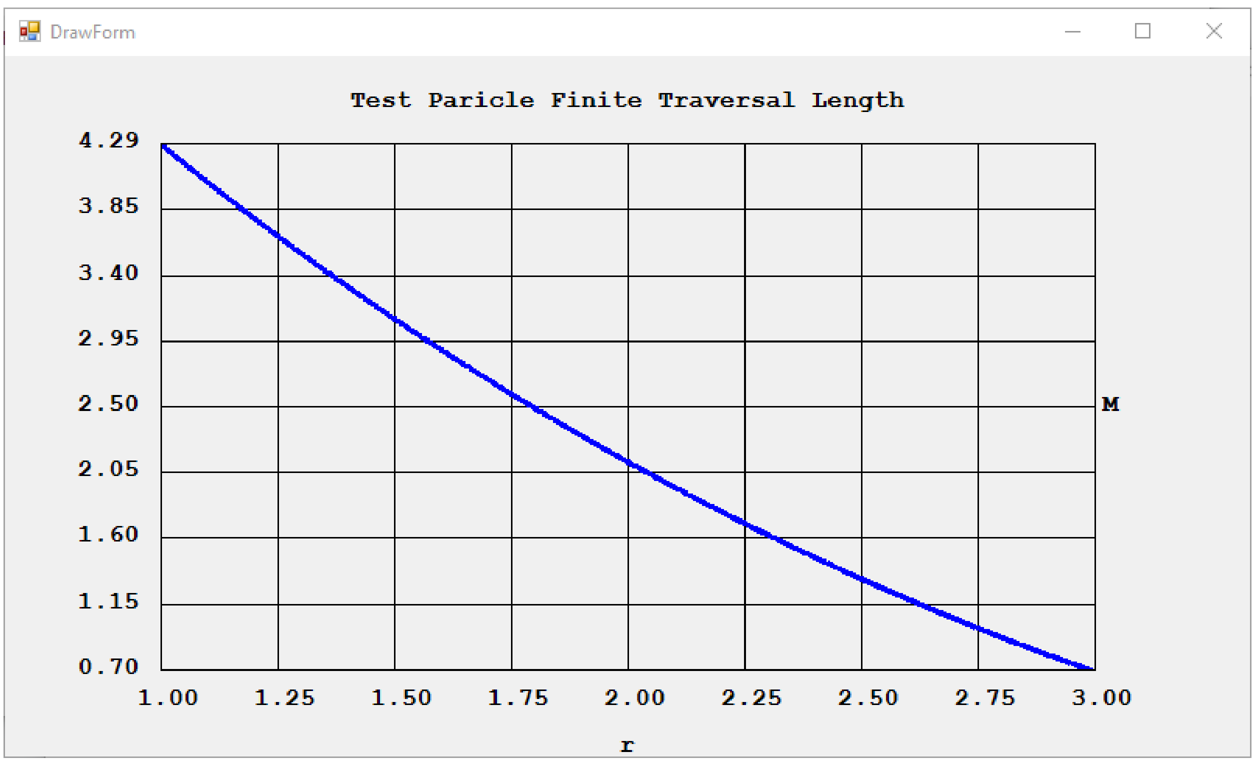
First the test particle finite distance:

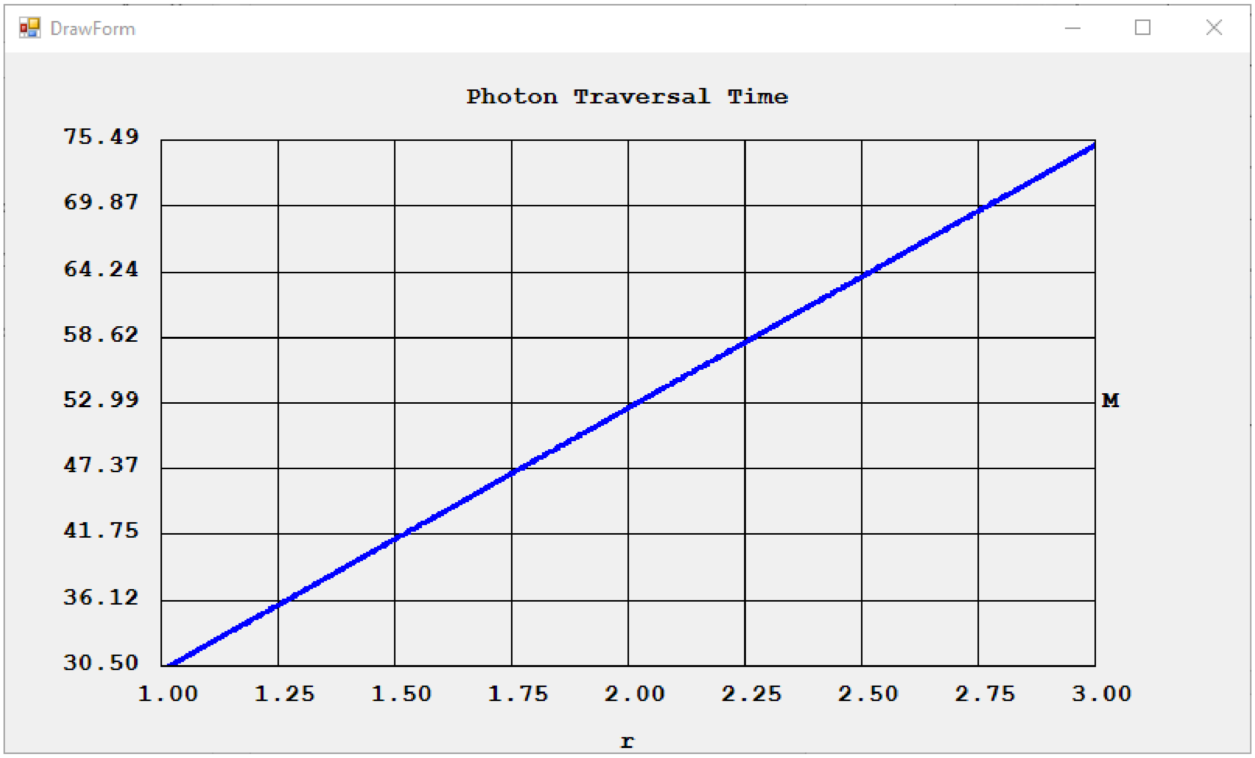
$$L_0 = \int_{r_0}^{2M} \frac{dr}{\sqrt{1 - 2M/r}}$$

Second the photon traversal time:

$$T_0 = \frac{1}{c} \int_{r_0}^{2M} \frac{dr}{1 - 2M/r}$$

Reference "General Relativity an Introduction to the Theory of the Gravitational Field" by Hans Stephani © 1982 Chapter "Gravitational Collapse and Black Holes". The two preceding equations are singular for r = 0 and are therefore unsuitable to describe the motion near the singularity.





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Reference "General Relativity an Introduction to the Theory of the Gravitational Field" by Hans Stephani © 1982 Chapter "Introduction". The Lagrange operator in spherical polar coordinates is :

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\vartheta}^2 + r^2(\sin\vartheta)^2\dot{\varphi}^2\right)$$

$$\ddot{r} - r\dot{\vartheta}^2 - r(\sin\vartheta)^2\dot{\varphi}^2 = 0$$

$$\ddot{\vartheta} + \frac{2}{r}\dot{r}\vartheta^{\cdot} - \sin\vartheta\cos\vartheta\,\dot{\varphi}^2 = 0$$

$$\ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} + 2\cot\vartheta\dot{\varphi}\dot{\vartheta} = 0$$

The classical arclength is:

$$\frac{ds}{dt} = \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\vartheta}{dt}\right)^2 + r^2(\sin\vartheta)^2\left(\frac{d\varphi}{dt}\right)^2}$$

We now use an n-quadrature function translated from an ACM Algorithm FORTRAN function to compute the arclength. The initial and final values are:

$$r_0 = 0, \vartheta_0 = 0, \varphi_0 = 0$$

$$r_1 = 0.000005c, \vartheta_0 = \frac{\pi}{4} \varphi_0 = \frac{\pi}{4}, c = 670616629 \, miles \; per \; hour$$

$$\frac{ds}{dt} = 1.3617567015 \times 10^6$$

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desired absolute and relative error 1.0e-10

f#	integral	epsilon	number	err code
1	1.3617567015E+006	4.026243E-004	45	2

Next, we calculate some examples using a fifth order Runge-Kutta algorithm. Reference “Numerical Analysis an Algorithmic Approach” © 1980 by S. D. Conte and Carl de Boor for the fourth order equations:

$$x_1 = r$$

$$x_2 = \dot{r}$$

$$y_1 = \vartheta$$

$$y_2 = \dot{\vartheta}$$

$$z_1 = \varphi$$

$$z_2 = \dot{\varphi}$$

$$\dot{x}_1 = x_2$$

$$\dot{y}_1 = y_2$$

$$\dot{z}_1 = z_2$$

$$K_{1i} = hf_i(x_{1n}, x_{2n}, y_{1n}, y_{2n}, z_{1n}, z_{2n}) \forall i \in \{1,2, \dots ,6\}$$

$$K_{2i} = hf_i\left(t_n + \frac{h}{2}, x_{1n} + \frac{K_{1i}}{2}, x_{2n} + \frac{K_{1i}}{2}, y_{1n} + \frac{K_{1i}}{2}, y_{2n} + \frac{K_{1i}}{2}, z_{1n} + \frac{K_{1i}}{2}, z_{2n} + \frac{K_{1i}}{2}\right)$$

$$K_{3i} = hf_i\left(t_n + \frac{h}{2}, x_{1n} + \frac{K_{2i}}{2}, x_{2n} + \frac{K_{2i}}{2}, y_{1n} + \frac{K_{2i}}{2}, y_{2n} + \frac{K_{2i}}{2}, z_{1n} + \frac{K_{2i}}{2}, z_{2n} + \frac{K_{2i}}{2}\right)$$

$$K_{4i} = hf_i\left(t_n + \frac{h}{2}, x_{1n} + \frac{K_{4i}}{2}, x_{2n} + \frac{K_{4i}}{2}, y_{1n} + \frac{K_{4i}}{2}, y_{2n} + \frac{K_{4i}}{2}, z_{1n} + \frac{K_{4i}}{2}, z_{2n} + \frac{K_{4i}}{2}\right)$$

$$y_{i,n+1} = y_{i,n} + \frac{1}{6}(K_{1i} + 2K_{2i} + 2K_{3i} + K_{4i})$$

I extended the previous equations to a fifth order method. The first test example is from "Numerical Analysis an Algorithmic Approach" © 1980 by S. D. Conte and Carl de Boor. They use the highly optimized DVERK FORTRAN numerical library function. I borrowed and translated the DIFFSYS C code equation solver from "A Numerical Library in C for Scientists and Engineers" © 1994 by H. T. Lau, PhD.

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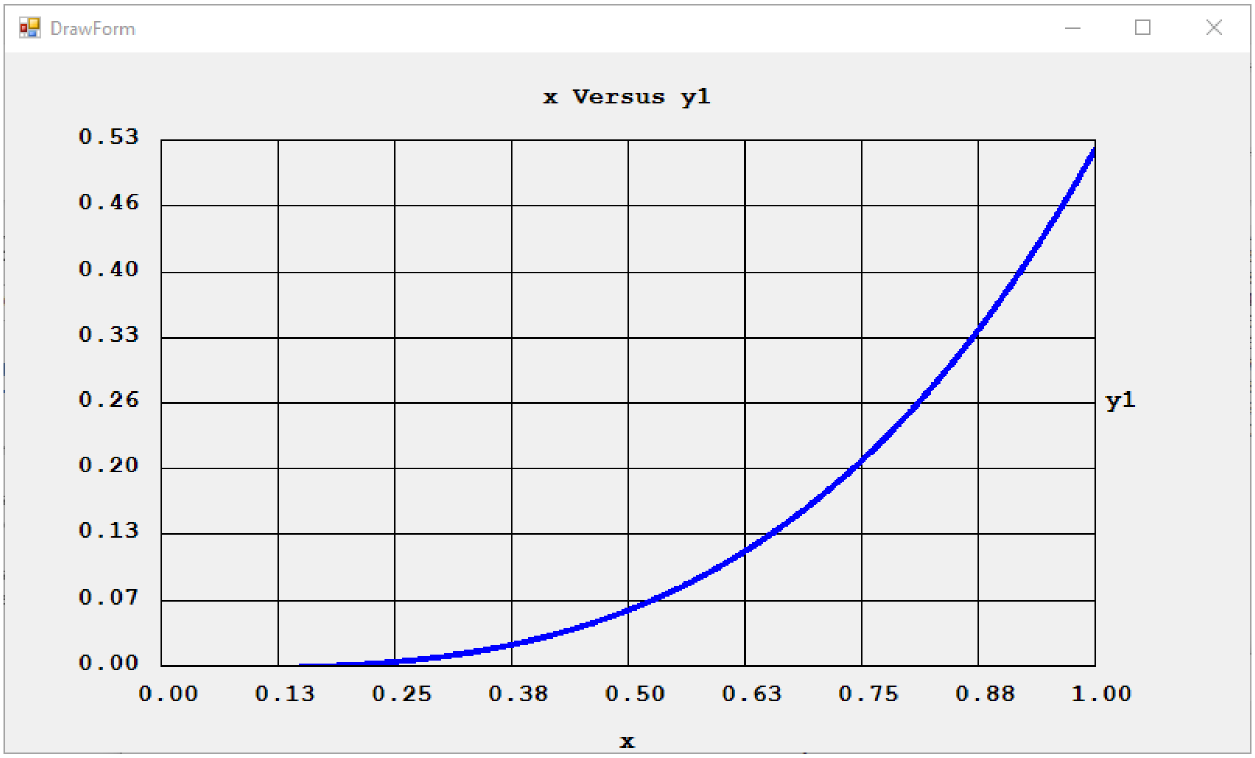
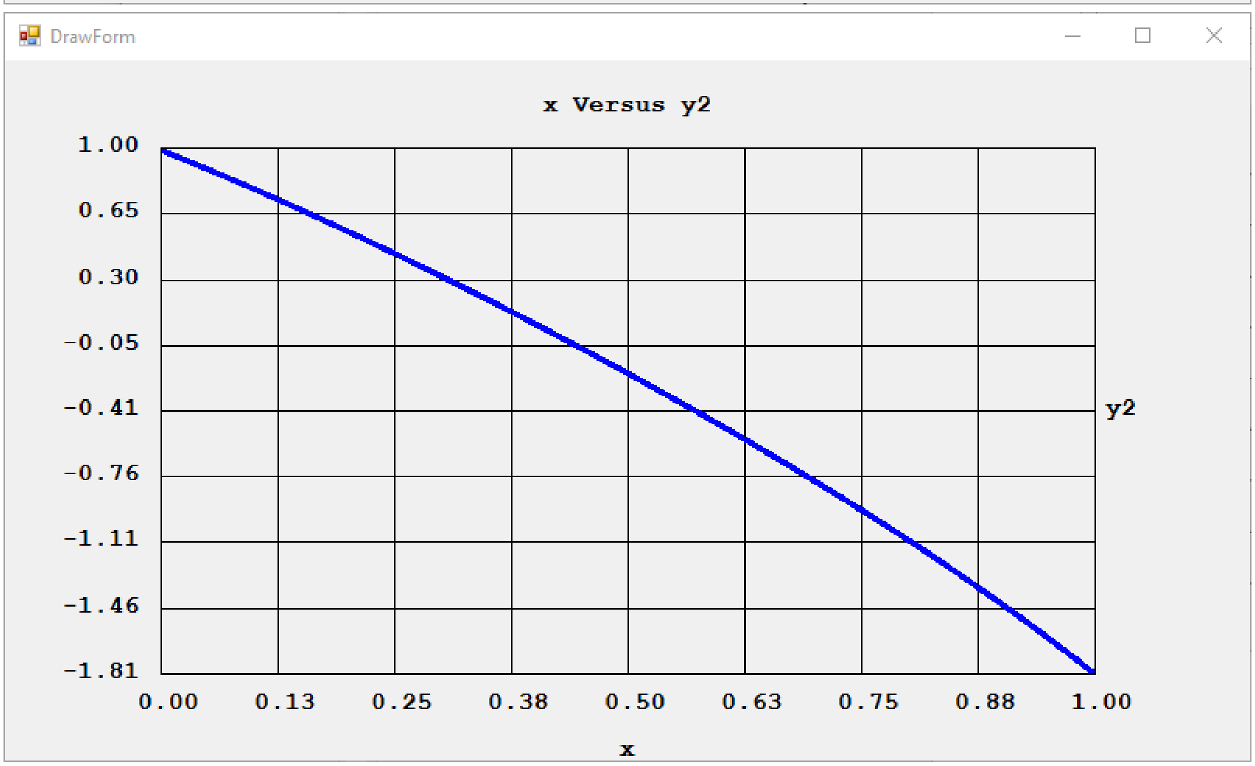
RK5

0.000	0.000000E+000	1.000000E+000	0.000000E+000	-2.000000E+000
0.100	4.475594E-004	7.905503E-001	1.353786E-002	-2.184036E+000
0.200	3.655435E-003	5.641419E-001	5.560178E-002	-2.340312E+000
0.300	1.258112E-002	3.231615E-001	1.282604E-001	-2.476988E+000
0.400	3.038157E-002	6.907606E-002	2.334562E-001	-2.604245E+000
0.500	6.040823E-002	-1.977543E-001	3.731697E-001	-2.734017E+000
0.600	1.062223E-001	-4.782483E-001	5.496636E-001	-2.879967E+000
0.700	1.716393E-001	-7.747889E-001	7.658355E-001	-3.057806E+000
0.800	2.608166E-001	-1.091462E+000	1.025721E+000	-3.286059E+000
0.900	3.784005E-001	-1.434401E+000	1.335208E+000	-3.587460E+000
1.000	5.297593E-001	-1.812315E+000	1.703054E+000	-3.991265E+000

Diffsys

0.000	0.000000E+000	1.000000E+000	0.000000E+000	-2.000000E+000
0.100	5.123423E-004	7.904769E-001	1.549221E-002	-2.185947E+000
0.200	4.195284E-003	5.635953E-001	6.388565E-002	-2.348434E+000
0.300	1.447960E-002	3.212831E-001	1.479831E-001	-2.496200E+000
0.400	3.507569E-002	6.448613E-002	2.706008E-001	-2.640107E+000
0.500	6.998423E-002	-2.070352E-001	4.347695E-001	-2.792991E+000
0.600	1.235320E-001	-4.949065E-001	6.440437E-001	-2.969793E+000
0.700	2.004460E-001	-8.023722E-001	9.029639E-001	-3.188069E+000
0.800	3.059838E-001	-1.134605E+000	1.217740E+000	-3.469062E+000
0.900	4.461473E-001	-1.499158E+000	1.597251E+000	-3.839547E+000
1.000	6.280191E-001	-1.906661E+000	2.054522E+000	-4.334869E+000





The second example is from the website:

https://www.math.utah.edu/~gustafso/2250systems-de.pdf

This is a brine tank cascade problem.

Rate of change = input rate – output rate

$$x_1' = -\frac{1}{2}x_1$$

$$x_2' = \frac{1}{2}x_1 - \frac{1}{4}x_2$$

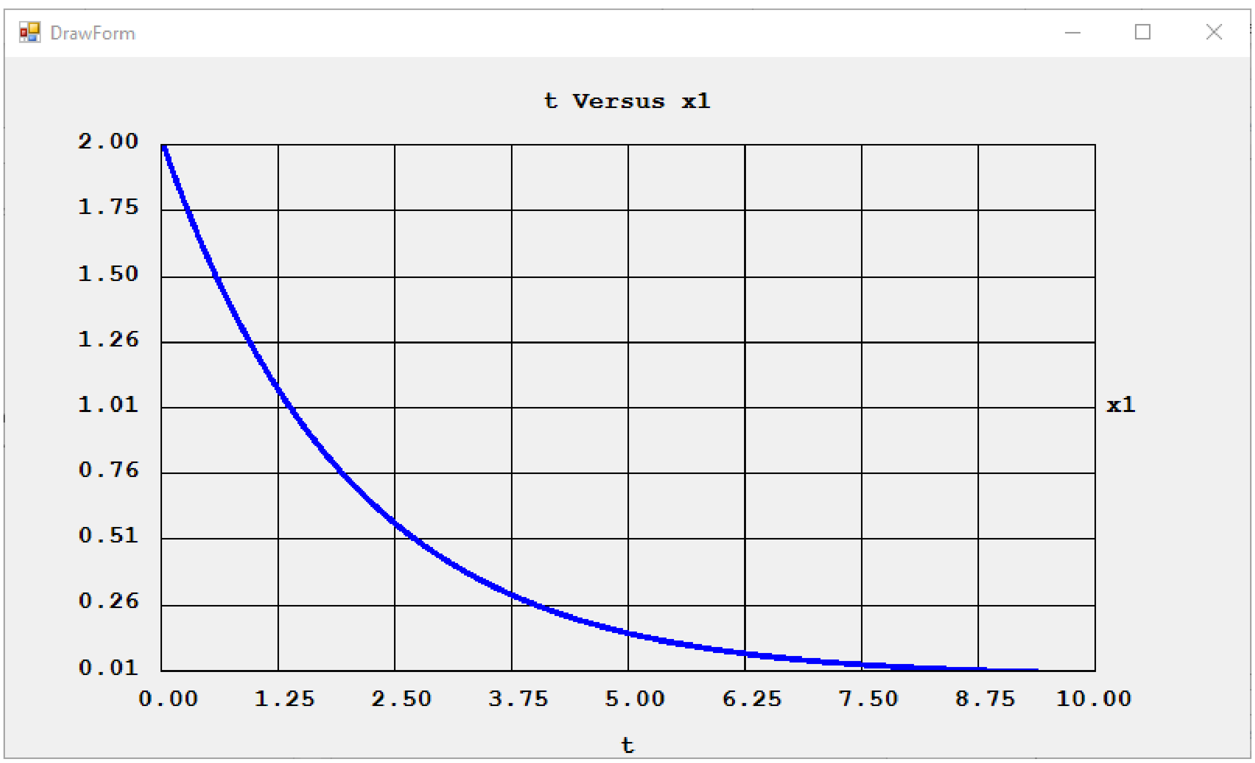
$$x_3' = \frac{1}{4}x_2 - \frac{1}{6}x_3$$

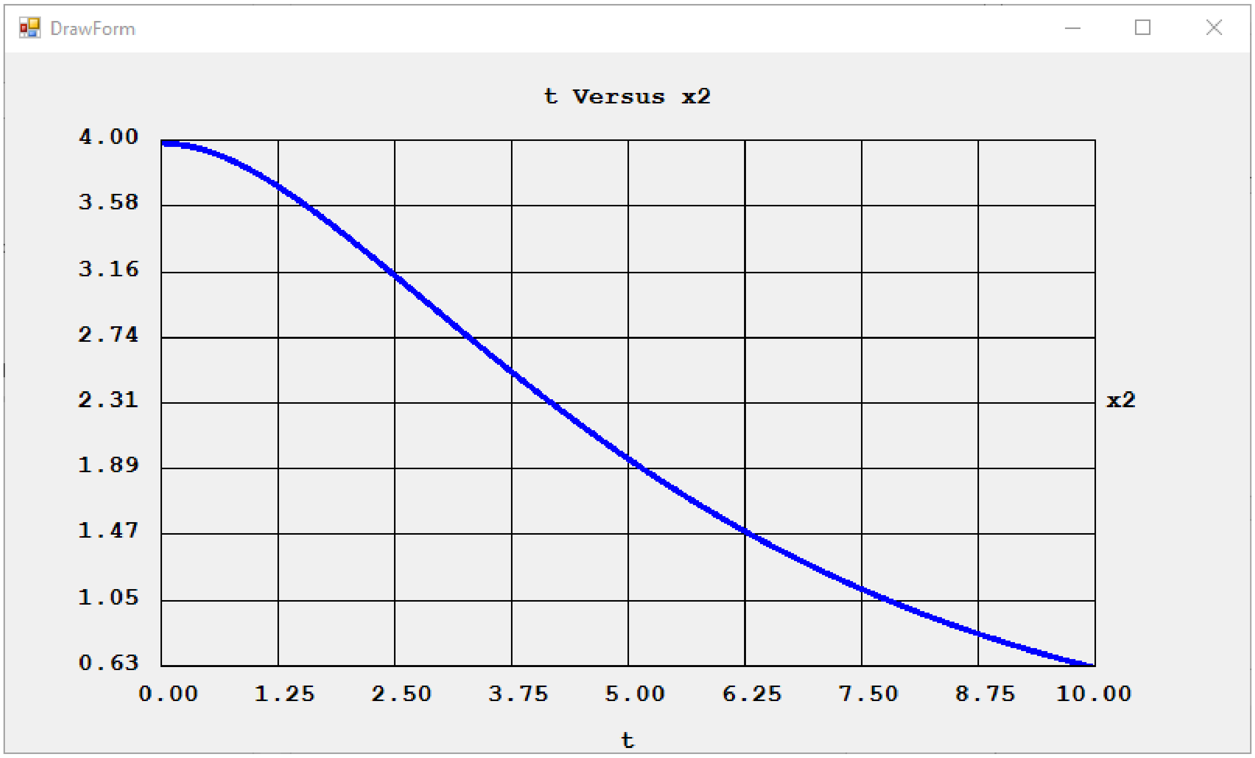
RK5

0.000	2.000000E+000	4.000000E+000	6.000000E+000
1.250	1.070260E+000	3.712047E+000	5.969375E+000
2.500	5.727284E-001	3.136111E+000	5.812920E+000
3.750	3.064842E-001	2.519302E+000	5.513797E+000
5.000	1.640089E-001	1.963460E+000	5.104457E+000
6.250	8.776611E-002	1.500846E+000	4.628605E+000
7.500	4.696628E-002	1.132457E+000	4.125650E+000
8.750	2.513307E-002	8.469246E-001	3.625772E+000
10.000	1.344946E-002	6.294597E-001	3.149631E+000

Diffsys

0.000	2.000000E+000	4.000000E+000	6.000000E+000
1.250	1.070523E+000	3.711879E+000	5.969291E+000
2.500	5.730096E-001	3.136072E+000	5.812737E+000
3.750	3.067099E-001	2.519425E+000	5.513588E+000
5.000	1.641700E-001	1.963698E+000	5.104292E+000
6.250	8.787387E-002	1.501143E+000	4.628522E+000
7.500	4.703549E-002	1.132769E+000	4.125664E+000
8.750	2.517628E-002	8.472226E-001	3.625878E+000
10.000	1.347589E-002	6.297282E-001	3.149815E+000
11.250	7.213126E-003	4.660111E-001	2.710092E+000
12.500	3.860908E-003	3.437737E-001	2.313196E+000

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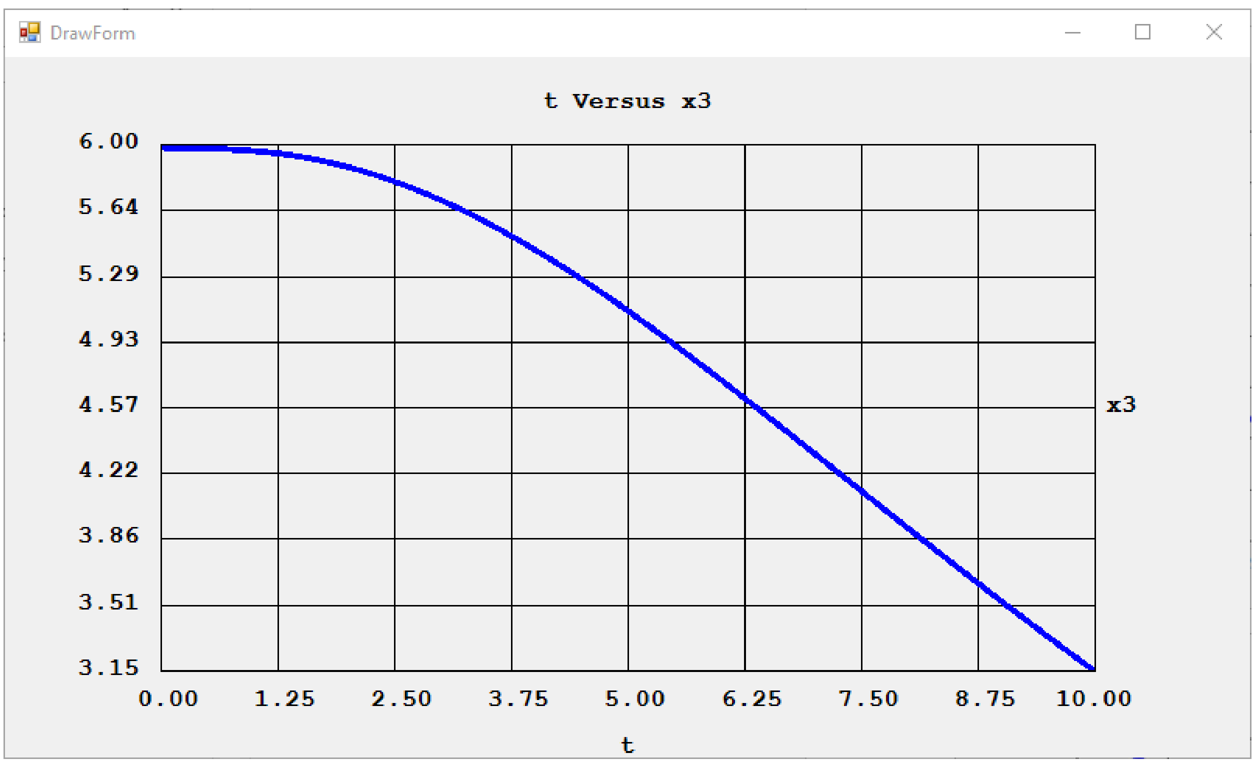
t Versus x2
4.00
3.58
3.16
2.74
2.31
1.89
1.47
1.05
0.63
0.00
1.25
2.50
3.75
5.00
6.25
7.50
8.75
10.00
t
x2


DrawForm
t Versus x3
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5.64
5.29
4.93
4.57
4.22
3.86
3.51
3.15
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2.50
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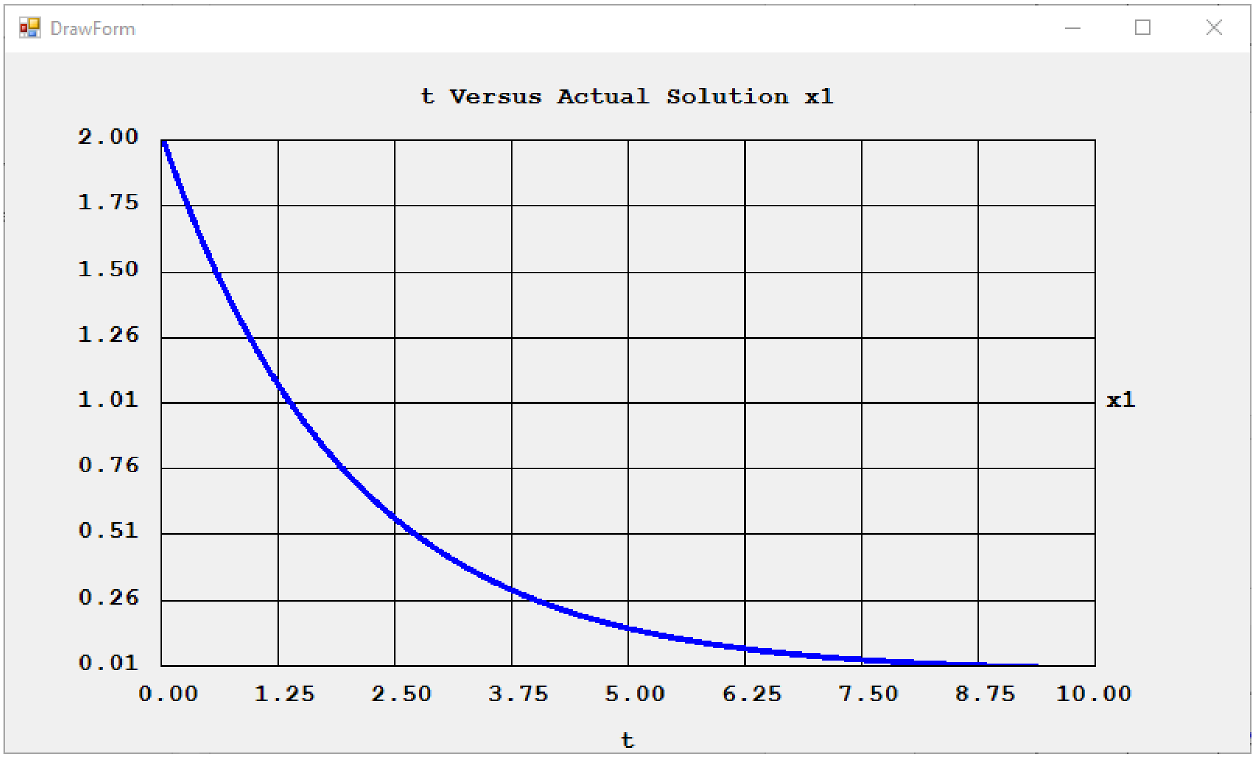
6.25
7.50
8.75
10.00
t
x3


DrawForm
t Versus Actual Solution x1
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1.75
1.50
1.26
1.01
0.76
0.51
0.26
0.01
x1
0.00
1.25
2.50
3.75
5.00
6.25
7.50
8.75
10.00
t