Christoffel Symbols of the Second Kind in Parabolic Coordinates © Friday, November 29, 2024, by James Pate Williams, Jr.

Online reference: [Parabolic coordinates - Wikipedia]

The definition of the Christoffel symbols of the second kind is given below:

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right) = \frac{1}{2} g^{im} \left(g_{mk,l} + g_{ml,k} + g_{kl,m} \right)$$

Where g is the metric tensor. The parabolic coordinates are:

$$x = \sigma\tau \cos\varphi$$

$$y = \sigma\tau \sin\varphi$$

$$z = \frac{1}{2}(\tau^2 - \sigma^2)$$

The arclength in Cartesian coordinates is:

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$dx = \tau \cos\varphi\, d\sigma + \sigma \cos\varphi\, d\tau - \sigma\tau \sin\varphi\, d\varphi$$

$$dy = \tau \sin\varphi\, d\sigma + \sigma \sin\varphi\, d\tau + \sigma\tau \cos\varphi\, d\varphi$$

$$dz = \tau d\tau - \sigma d\sigma$$

$$dx^2 = \tau^2(\cos\varphi)^2 d\sigma^2 + \sigma^2(\cos\varphi)^2 d\tau^2 + \sigma^2\tau^2(\sin\varphi)^2 d\varphi^2 + \cdots$$

$$dy^2 = \tau^2(\sin\varphi)^2 d\sigma^2 + \sigma^2(\sin\varphi)^2 d\tau^2 + \sigma^2\tau^2(\cos\varphi)^2 d\varphi^2 + \cdots$$

$$dz^2 = \tau^2 d\tau^2 + \sigma^2 d\sigma^2 + \cdots$$

The arclength in parabolic coordinates is:

$$\begin{aligned} ds^2 &= \tau^2 d\sigma^2 + \sigma^2 d\tau^2 + \sigma^2\tau^2 d\varphi^2 + \tau^2 d\tau^2 + \sigma^2 d\sigma^2 \\ &= (\tau^2 + \sigma^2) d\tau^2 + (\tau^2 + \sigma^2) d\sigma^2 + \sigma^2\tau^2 d\varphi^2 \end{aligned}$$

The three nonzero metric tensor elements are:

$$g_{\tau\tau} = \tau^2 + \sigma^2$$

$$g_{\sigma\sigma} = \tau^2 + \sigma^2$$

$$g_{\varphi\varphi} = \sigma^2\tau^2$$

The six nonzero metric tensor partial derivatives are:

$$g_{\tau\tau,\tau} = 2\tau$$

$$g_{\tau\tau,\sigma} = 2\sigma$$

$$g_{\sigma\sigma,\tau} = 2\tau$$

$$g_{\sigma\sigma,\sigma} = 2\sigma$$

$$g_{\varphi\varphi,\tau} = 2\tau\sigma^2$$

$$g_{\varphi\varphi,\sigma} = 2\tau^2\sigma$$

An enumeration of the possible twenty-seven numeric indices of the Christoffel symbols of the second kind is given by:

111,112,113,121,122,123,131,132,133,

211,212,213,221,222,223,231,232,233,

311,312,313,321,322,323,331,332,333

Where:

$$\tau = 1, \sigma = 2, \varphi = 3$$

$$\Gamma^{\tau}_{\tau\tau}, \Gamma^{\tau}_{\tau\sigma}, \Gamma^{\tau}_{\tau\varphi}, \Gamma^{\tau}_{\sigma\tau}, \Gamma^{\tau}_{\sigma\sigma}, \Gamma^{\tau}_{\sigma\varphi}, \Gamma^{\tau}_{\varphi\tau}, \Gamma^{\tau}_{\varphi\varphi,}$$

$$\Gamma^{\sigma}_{\tau\tau}, \Gamma^{\sigma}_{\tau\sigma}, \Gamma^{\sigma}_{\tau\varphi}, \Gamma^{\sigma}_{\sigma\tau}, \Gamma^{\sigma}_{\sigma\sigma}, \Gamma^{\sigma}_{\sigma\varphi}, \Gamma^{\sigma}_{\varphi\tau}, \Gamma^{\sigma}_{\varphi\sigma}, \Gamma^{\sigma}_{\varphi\varphi},$$

$$\Gamma^{\varphi}_{\tau\tau}, \Gamma^{\varphi}_{\tau\sigma}, \Gamma^{\varphi}_{\tau\varphi}, \Gamma^{\varphi}_{\sigma\tau}, \Gamma^{\varphi}_{\sigma\sigma}, \Gamma^{\varphi}_{\sigma\varphi}, \Gamma^{\varphi}_{\varphi\tau}, \Gamma^{\varphi}_{\varphi\sigma}, \Gamma^{\varphi}_{\varphi\varphi}$$

$$g^{\tau\tau} = g^{\sigma\sigma} = \frac{1}{\tau^2 + \sigma^2}$$

$$g^{\varphi\varphi} = \frac{1}{\tau^2\sigma^2}$$

The twenty-seven Christoffel symbols of the second kind are given by:

$$\Gamma^{\tau}_{\tau\tau} = \frac{1}{2} g^{\tau\tau}\left(g_{\tau\tau,\tau} + g_{\tau\tau,\tau} - g_{\tau\tau,\tau}\right) = \frac{2\tau}{2(\tau^2 + \sigma^2)} = \frac{\tau}{\tau^2 + \sigma^2}, (1)$$

$$\Gamma^{\tau}_{\tau\sigma} = \Gamma^{\tau}_{\sigma\tau} = \frac{1}{2} g^{\tau\tau}\left(g_{\tau\tau,\sigma} + g_{\tau\sigma,\tau} - g_{\tau\sigma,\tau}\right) = \frac{\sigma}{\tau^2 + \sigma^2}, (2,3)$$

$$\Gamma^{\tau}_{\tau\varphi} = \Gamma^{\tau}_{\varphi\tau} = \frac{1}{2} g^{\tau\tau}\left(g_{\tau\tau,\varphi} + g_{\tau\varphi,\tau} - g_{\tau\varphi,\tau}\right) = 0, (4,5)$$

$$\Gamma^{\sigma}_{\sigma\sigma} = \frac{\sigma}{\tau^2 + \sigma^2}, (6)$$

$$\Gamma^{\tau}_{\varphi\varphi} = \Gamma^{\sigma}_{\varphi\varphi} = \Gamma^{\varphi}_{\varphi\varphi} = 0, (7,8,9)$$

$$\Gamma^{\tau}_{\sigma\sigma} = \Gamma^{\sigma}_{\tau\tau} = \Gamma^{\varphi}_{\tau\tau} = \Gamma^{\varphi}_{\sigma\sigma} = 0, (10,11,12,13)$$

$$\Gamma^{\sigma}_{\sigma\tau} = \Gamma^{\sigma}_{\tau\sigma} = \frac{\tau}{\tau^2 + \sigma^2} (14,15)$$

$$\Gamma^{\tau}_{\sigma\varphi} = \Gamma^{\tau}_{\varphi\sigma} = \frac{1}{2} g^{\tau\tau}\left(g_{\tau\sigma,\varphi} + g_{\tau\varphi,\sigma} - g_{\sigma\varphi,\tau}\right) = 0, (16,17)$$

$$\Gamma^{\tau}_{\varphi\varphi} = \frac{1}{2} g^{\tau\tau}\left(g_{\tau\varphi,\varphi} + g_{\tau\varphi,\varphi} - g_{\varphi\varphi,\tau}\right) = -\frac{\tau\sigma^2}{\tau^2 + \sigma^2}, (18)$$

$$\Gamma^{\sigma}_{\tau\varphi} = \Gamma^{\sigma}_{\varphi\tau} = \frac{1}{2} g^{\sigma\sigma}\left(g_{\sigma\tau,\varphi} + g_{\sigma\varphi,\tau} - g_{\tau\varphi,\sigma}\right) = 0, (19,20)$$

$$\Gamma^{\sigma}_{\sigma\varphi} = \Gamma^{\sigma}_{\varphi\sigma} = \frac{1}{2} g^{\sigma\sigma}\left(g_{\sigma\sigma,\varphi} + g_{\sigma\varphi,\sigma} - g_{\sigma\varphi,\sigma}\right) = 0, (21.22)$$

$$\Gamma^{\varphi}_{\tau\sigma} = \Gamma^{\varphi}_{\sigma\tau} = \frac{1}{2} g^{\varphi\varphi}\left(g_{\varphi\tau.\sigma} + g_{\varphi\sigma,\tau} - g_{\tau\sigma,\varphi}\right) = 0, (23,24)$$

$$\Gamma^{\varphi}_{\tau\varphi} = \Gamma^{\varphi}_{\varphi\tau} = \frac{1}{2} g^{\varphi\varphi}\left(g_{\varphi\varphi,\tau} + g_{\varphi\tau,\varphi} - g_{\varphi\tau,\varphi}\right) = \frac{\tau\sigma^2}{\tau^2\sigma^2} = \frac{1}{\tau}, (25)$$

$$\Gamma^{\varphi}_{\sigma\varphi} = \Gamma^{\varphi}_{\varphi\sigma} = \frac{1}{2} g^{\varphi\varphi}\left(g_{\varphi\sigma,\varphi} + g_{\varphi\varphi,\sigma} - g_{\sigma\varphi,\varphi}\right) = \frac{2\tau^2\sigma}{2\sigma^2\tau^2} = \frac{1}{\sigma}, (26)$$

$$\Gamma^{\varphi}_{\varphi\varphi} = \frac{1}{2} g^{\varphi\varphi}\left(g_{\varphi\varphi,\varphi} + g_{\varphi\varphi,\varphi} - g_{\varphi\varphi,\varphi}\right) = 0, (27)$$