

Problems from the Textbook: **Mathematical Methods in the Physical Sciences Second Edition** © 1983 by Mary L. Boas, Solutions Provided by James Pate Williams, Jr.

PROBLEMS, SECTION 2

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$$

$$\sum_{n=1}^{\infty} \frac{n}{n+5} = \frac{1}{6} + \frac{2}{7} + \frac{3}{8} + \frac{4}{9} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1} = \frac{1}{2} + \frac{\sqrt{2}}{3} + \frac{\sqrt{3}}{4} + \frac{\sqrt{4}}{5} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{2n(2n+1)}{3n+5} = \frac{2 \cdot 3}{8} + \frac{4 \cdot 5}{11} + \frac{6 \cdot 7}{14} + \frac{8 \cdot 9}{17} + \cdots = \frac{6}{8} + \frac{20}{11} + \frac{42}{14} + \frac{72}{17} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} = \frac{1}{2!} + \frac{(2)^2}{4!} + \frac{(6)^2}{6!} + \frac{(24)^2}{8!} + \cdots = \frac{1}{2} + \frac{4}{24} + \frac{36}{720} + \frac{576}{40,320} + \cdots$$

$$\frac{1}{3} + \frac{2}{5} + \frac{4}{7} + \frac{8}{9} + \frac{16}{11} + \cdots = \sum_{n=0}^{\infty} \frac{2^n}{2n+3}$$

$$\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} + \cdots = \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$$

$$\frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^2}$$

$$\frac{1}{7} + \frac{2}{9} + \frac{3}{11} + \frac{4}{13} + \cdots = \sum_{n=1}^{\infty} \frac{n}{2n+5}$$

$$\frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}}$$

$$\frac{\ln 2}{2} - \frac{\ln 3}{3} + \frac{\ln 4}{4} - \frac{\ln 5}{5} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ln(n+1)}{n+1}$$

PROBLEMS, SECTION 4

$$S = \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$|S - S_4| = \left| S - \sum_{n=1}^4 \frac{1}{2^n} \right| = \left| 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} \right| = |1 - 0.9375| = 0.0625$$

$$|S - S_5| = \left| 1 - 0.9375 - \frac{1}{32} \right| = |0.0625 - 0.03125| = 0.03125$$

$$|S - S_6| = \left| 0.03125 - \frac{1}{64} \right| = 0.016525$$

$$|S - S_7| = \left| 0.01625 - \frac{1}{128} \right| = 0.0084375$$

$$S = \sum_{n=1}^{\infty} \frac{1}{5^n} < \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$|S - S_4| = \left| S - \sum_{n=1}^4 \frac{1}{5^n} \right| = \left| 0.25 - \frac{1}{5} - \frac{1}{25} - \frac{1}{125} - \frac{1}{625} \right| = |0.25 - 0.2496| = 0.0004$$

$$|S - S_5| = \left| 0.0004 - \frac{1}{3,125} \right| = 0.00008$$

$$S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots$$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} < \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$|S - S_4| = |1 - 0.883333 \dots| = 0.1666 \dots$$

$$S = \sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$$

$$\frac{1}{n!} < \frac{1}{2^n} \forall n \geq 3$$

$$S = \sum_{n=1}^{\infty} \frac{1}{n2^n} = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \dots$$

$$\frac{1}{n2^n} < \frac{1}{2^n} \forall n \geq 2$$

$$S = \sum_{n=1}^{\infty} \frac{1}{2^n + 3^n} = \frac{1}{2+3} + \frac{1}{4+9} + \frac{1}{8+27} + \frac{1}{16+81} + \dots = \frac{1}{5} + \frac{1}{13} + \frac{1}{35} + \frac{1}{97} + \dots$$

$$2^n + 3^n > 2^n \forall n \geq 1$$

$$\therefore \frac{1}{2^n + 3^n} < \frac{1}{2^n} \forall n \geq 1$$

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \dots$$

$$T = \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \dots$$

$$\therefore \frac{1}{n^2} \geq \frac{1}{2^n}$$

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$$

$$\frac{1}{(n+1)^2} < \frac{1}{n(n+1)}$$

$$S = \sum_{n=2}^{\infty} \frac{1}{n^2 - 1} = \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \dots$$

$$\frac{1}{n^2 - 1} = \frac{1}{(n+1)(n-1)} \forall n \geq 2$$

PROBLEMS, SECTION 5

$$\frac{1}{2} - \frac{4}{5} + \frac{9}{10} - \frac{16}{17} + \frac{25}{26} - \frac{36}{37} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2 + 1}$$

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} n^2}{n^2 + 1} \right| = \lim_{n \rightarrow \infty} \left(\frac{2n}{2n} \right) = 1 \therefore \text{the series diverges}$$

$$\sqrt{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{4}}{3} + \frac{\sqrt{5}}{4} + \frac{\sqrt{6}}{5} + \dots = \sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n}$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n+1}}{n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{2\sqrt{n+1}}}{1} = 0$$

$$\sum_{n=1}^{\infty} \frac{n+3}{n^2+10n}$$

$$\lim_{n \rightarrow \infty} \frac{n+3}{n^2+10n} = \lim_{n \rightarrow \infty} \frac{1}{2n+10} = 0$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{(n+1)^2}$$

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^n n^2}{(n+1)^2} \right| = \frac{2n}{2n+2} = \frac{2}{2} = 1 \therefore \text{the series diverges}$$

$$\sum_{n=1}^{\infty} \frac{n!}{n!+1}$$

$$\lim_{n \rightarrow \infty} \frac{n!}{n!+1} \approx \lim_{n \rightarrow \infty} \frac{n \ln n - n}{n \ln n - n + 1} = \frac{\ln n + 1 - 1}{\ln n + 1 - 1} = 1 \therefore \text{the series diverges}$$

$$\sum_{n=1}^{\infty} \frac{n!}{(n+1)!}$$

$$\lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} \approx \lim_{n \rightarrow \infty} \frac{n \ln n - n}{(n+1) \ln(n+1) - (n+1)} = \lim_{n \rightarrow \infty} \frac{\ln n + 1 - 1}{\ln(n+1) + 1 - 1} = \lim_{n \rightarrow \infty} \frac{\ln n}{\ln(n+1)}$$

$$= \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n \rightarrow \infty} \frac{n+1}{n} = 1 \therefore \text{the series diverges}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^3+1}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^n n}{\sqrt{n^3 + 1}} \right| = \lim_{n \rightarrow \infty} n(n^3 + 1)^{-1/2} = \lim_{n \rightarrow \infty} \left| (n^3 + 1)^{-1/2} - \frac{1}{2} \cdot 3n^3(n^3 + 1)^{-3/2} \right| > 0$$

$\therefore$ the series diverges

$$\sum_{n=1}^{\infty} \frac{\ln n}{n}$$

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{1} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

$$\sum_{n=1}^{\infty} \frac{3^n}{2^n + 3^n}$$

$$\lim_{n \rightarrow \infty} \frac{3^n}{2^n + 3^n} = \lim_{n \rightarrow \infty} \frac{3^n \ln 3}{2^n \ln 2 + 3^n \ln 3} = 0$$

$$\sum_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) = \sum_{n=2}^{\infty} \frac{n^2 - 1}{n^2}$$

$$\lim_{n \rightarrow \infty} \frac{n^2 - 1}{n^2} = \lim_{n \rightarrow \infty} \frac{2n}{2n} = 1 \therefore \text{the series diverges}$$

PROBLEMS, SECTION 6

1. Show that

$$n! > 2^n \forall n \geq 4$$

$$4! = 24 > 2^4 = 16$$

$$5! = 120 > 2^5 = 32$$

$$6! = 720 > 2^6 = 64$$

2.

$$1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \frac{8}{2^4} + \dots = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

$$\lim_{n \rightarrow \infty} \frac{2^{n-1}}{2^n} = \frac{1}{2} > 0 \therefore \text{the series diverges}$$

$$\frac{1}{n} < \frac{2^{n-1}}{2^n} = \frac{1}{2} \forall n \geq 3 \therefore \sum_{n=1}^{\infty} \frac{1}{n} \therefore \text{the harmonic series diverges}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots < 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

$$\frac{1}{n^2} < \frac{1}{2^{n-1}} \forall n \geq 2 \therefore \text{the series diverges}$$

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} < \sum_{n=1}^{\infty} \frac{1}{2^n} \therefore \text{the series diverges}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 3^n} < \sum_{n=1}^{\infty} \frac{1}{2^n} \therefore \text{the series diverges}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} < \sum_{n=1}^{\infty} \frac{1}{n} \therefore \text{the series diverges}$$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{1}{\frac{1}{2\sqrt{n}}} > 0 \therefore \text{the series diverges}$$

$$\sum_{n=1}^{\infty} \frac{1}{\ln n} < \sum_{n=1}^{\infty} \frac{1}{n} \therefore \text{the series diverges}$$

$$\lim_{n \rightarrow \infty} \frac{1}{\ln n} = n \neq 0 \therefore \text{the series diverges}$$

There are 9 one-digit numbers (1-9), 90 two-digit numbers (10-99). How many three-digit numbers, four-digit numbers, etc. numbers are there?

$$\frac{9}{10} + \frac{90}{100} + \frac{900}{1000} + \frac{9000}{10,000} + \cdots = \frac{9}{10} + \frac{9}{10} + \frac{9}{10} + \frac{9}{10} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

$$\frac{1}{n} < \frac{9}{10} \forall n \geq 2, \therefore \text{the harmonic series diverges}$$

PROBLEMS, SECTION 6 Continued

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

$$\int \frac{dn}{n \ln n} < \int \frac{dn}{n}$$

$$\int \frac{dn}{n \ln n} = \int \frac{dn}{\ln n^n}$$

$$\sum_{n=1}^{\infty} \frac{n}{n^2 + 4}$$

$$\lim_{n \rightarrow \infty} \frac{n}{n^2 + 4} = \lim_{n \rightarrow \infty} \frac{1}{2n} = 0$$

$$\int \frac{ndn}{n^2 + 4} = \frac{1}{2} \int d \ln(n^2 + 4) = \frac{1}{2} \ln(n^2 + 4) + C \therefore \text{the series diverges}$$

$$\sum_{n=3}^{\infty} \frac{1}{n^2 - 4}$$

$$\frac{1}{n^2 - 4} = \frac{1}{(n - 2)(n + 2)} = \frac{A}{n - 2} + \frac{B}{n + 2}$$

$$\begin{aligned} \int \frac{dn}{n^2 - 4} &= \int \frac{dn}{(n - 2)(n + 2)} = \int \frac{Adn}{n - 2} \\ &+ \int \frac{Bdn}{n + 2} = A \ln(n - 2) + B \ln(n + 2) + C \therefore \text{the series diverges} \end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{e^n}{e^{2n} + 9}$$

$$\begin{aligned} \int \frac{e^n dn}{e^{2n} + 9} &= \int \frac{e^n dn}{(e^n - 3)(e^n + 3)} \\ &= \int \frac{Ae^n dn}{e^n - 3} \\ &+ \int \frac{Be^n dn}{e^n + 3} = A \ln(e^n - 3) + B \ln(e^n + 3) + C \therefore \text{the series diverges} \end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)^{3/2}}$$

$$\int \frac{dn}{n(1 + \ln n)^{3/2}} = \int d(1 + \ln n)^{1/2} = (1 + \ln n)^{1/2} + C \rightarrow \text{the series diverges}$$

$$\sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)^2}$$

$$\int \frac{ndn}{(n^2 + 1)^2} = \int n(n^2 + 1)^{-1}dn = -\frac{1}{2} \int d(n^2 + 1)^{-1}dn = -\frac{1}{2}(n^2 + 1)^{-1} + C$$

$\rightarrow$ the series converges

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$$

$$\int \frac{n^2 dn}{n^3 + 1} = \frac{1}{3} \ln(n^3 + 1) + C \rightarrow \infty \therefore \text{the series diverges}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 9}}$$

$$\int \frac{dn}{\sqrt{n^2 + 9}} = \int \frac{dn}{\sqrt{(n-3)(n+3)}}$$

$$\frac{1}{\sqrt{n^2 + 9}} = \frac{1}{\sqrt{9 \left[\left(\frac{n}{3} \right)^2 + 1 \right]}} = \frac{1}{3 \sqrt{\left(\frac{n}{3} \right)^2 + 1}}$$

$$\int \frac{dn}{\sqrt{n^2 + 9}} = \frac{1}{3} \int \left[\left(\frac{n}{3} \right)^2 + 1 \right]^{1/2} dn = \int (m^2 + 1)^{-1/2} dm$$

$$m = \tan x$$

$$(\cos x)^2 + (\sin x)^2 = 1$$

$$(\cos x)^2 = 1 - (\sin x)^2$$

$$1 = \frac{1}{(\cos x)^2} - (\tan x)^2 = (\sec x)^2 - (\tan x)^2$$

$$(\sec x)^2 = 1 + (\tan x)^2$$

$$dm = d \tan x = d \left(\frac{\sin x}{\cos x} \right) = \left[\frac{\cos x}{\cos x} + (\tan x)^2 \right] dx = [1 + (\tan x)^2] dx = (\sec x)^2 dx$$

$$\int (m^2 + 1)^{-1/2} dm = \int \sec x dx = \int \frac{dx}{\cos x} = \tan x \sec x + C$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

Suppose $p = 1$ then we have the divergent harmonic series by the integral test:

$$\int \frac{dn}{n} = \ln n + C$$

$$\int \frac{dn}{n^p} = \int n^{-p} dn = -\frac{n^{-p+1}}{(1-p)} + C \therefore \text{converges for } p > 1, \text{ otherwise diverges}$$

$$\sum_{n=1}^{\infty} e^{-n^2}$$

$$\int \frac{dn}{e^{n^2}} < \int \frac{dn}{e^n} = -e^{-n} \ln e \rightarrow 0 \text{ as } n \rightarrow \infty$$

PROBLEMS, SECTION 6 Continued

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2}$$

$$\rho_n = \frac{\frac{2^{n+1}}{(n+1)^2}}{\frac{2^n}{n^2}} = \frac{n^2 2^{n+1}}{2^n (n+1)^2} = \frac{2n^2}{(n+1)^2}$$

$$\rho = \lim_{n \rightarrow \infty} \frac{2n^2}{(n+1)^2} = \lim_{n \rightarrow \infty} \frac{4n}{2n+1} = \frac{4}{2} = 2 \therefore \text{the series diverges}$$

$$\sum_{n=1}^{\infty} \frac{3^n}{2^{2n}}$$

$$\rho_n = \frac{\frac{3^{n+1}}{2^{2n+2}}}{\frac{3^n}{2^{2n}}} = \frac{2^{2n} 3^{n+1}}{2^{2n+2} 3^n} = \frac{3}{2^2} = \frac{3}{4}$$

$$\rho = \lim_{n \rightarrow \infty} \frac{3}{4} = \frac{3}{4} < 1 \therefore \text{the series converges}$$

$$\sum_{n=0}^{\infty} \frac{n!}{(2n)!}$$

$$\rho_n = \frac{\frac{(n+1)!}{[2(n+1)]!}}{\frac{n!}{(2n)!}} = \frac{(2n)!(n+1)!}{n! [2(n+1)]!}$$

$$\rho_2 = \frac{4! 3!}{2! 6!} = \frac{4 \cdot 3 \cdot 6}{720} = \frac{12 \cdot 6}{720} = \frac{72}{720} < 1$$

$$\rho = \lim_{n \rightarrow \infty} \frac{(2n)! (n+1)!}{n! [2(n+1)]!} < 1 \therefore \text{the series converges}$$

$$\sum_{n=0}^{\infty} \frac{5^n (n!)^2}{(2n)!}$$

$$\rho_n = \frac{\frac{5^{n+1} [(2n+2)!]^2}{[2(n+1)]!}}{\frac{5^n (n!)^2}{(2n)!}} = \frac{(2n)! 5^{n+1} [(2n+2)!]^2}{5^n (n!)^2 [2(n+1)]!} = \frac{(2n)! [(2n+2)!]^2}{(n!)^2 [2(n+1)]!}$$

$$\rho > 1 \therefore \text{the series diverges}$$

$$\sum_{n=1}^{\infty} \frac{10^n}{(n!)^2}$$

$$\rho_n = \frac{\frac{10^{n+1}}{[(n+1)!]^2}}{\frac{10^n}{(n!)^2}} = \frac{10^{n+1} (n!)^2}{10^n [(n+1)!]^2} = \frac{10 (n!)^2}{[(n+1)!]^2}$$

$$\rho_2 = \frac{10 \cdot 4}{36} = \frac{40}{36}$$

$$\rho_3 = \frac{10 \cdot 36}{576} = \frac{360}{576}$$

$$\rho = \lim_{n \rightarrow \infty} \frac{10 (n!)^2}{[(n+1)!]^2} < 1 \therefore \text{the series converges}$$

$$\sum_{n=1}^{\infty} \frac{n!}{100^n}$$

$$\rho_n = \frac{\frac{(n+1)!}{100^{n+1}}}{\frac{n!}{100^n}} = \frac{100^n (n+1)!}{100^{n+1} n!} \approx \frac{(n+1) \ln(n+1) - (n+1)}{100(n \ln n - n)}$$

$$\rho = \lim_{n \rightarrow \infty} \frac{(n+1) \ln(n+1) - (n+1)}{100(n \ln n - n)} > 1 \therefore \text{the series diverges}$$

$$\sum_{n=0}^{\infty} \frac{3^{2n}}{2^{3n}}$$

$$\rho_n = \frac{\frac{3^{2n+2}}{2^{3n+3}}}{\frac{3^{2n}}{2^{3n}}} = \frac{3^{2n+2} 2^{3n}}{3^{2n} 2^{3n+3}} = \frac{9}{8}$$

$$\rho = \frac{9}{8} \therefore \text{the series diverges}$$

$$\sum_{n=1}^{\infty} \frac{e^n}{\sqrt{n!}}$$

$$\rho_n = \frac{\frac{e^{n+1}}{\sqrt{(n+1)!}}}{\frac{e^n}{\sqrt{n!}}} = \frac{e\sqrt{n!}}{\sqrt{(n+1)!}}$$

$$\rho = \lim_{n \rightarrow \infty} \frac{e\sqrt{n!}}{\sqrt{(n+1)!}} < 1 \therefore \text{the series converges}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^3 e^{3n}}{(3n)!}$$

$$\rho_n = \frac{\frac{[(n+1)!]^3 e^{3n+3}}{(3n+3)!}}{\frac{(n!)^3 e^{3n}}{(3n)!}} = \frac{(3n)! [(n+1)!]^3 e^{3n+3}}{(n!)^3 (3n+3)! e^{3n}}$$

$$\rho = \lim_{n \rightarrow \infty} \frac{(3n)! [(n+1)!]^3 e^3}{(n!)^3 (3n+3)!}$$

$$\lim_{n \rightarrow \infty} \frac{(3n)!}{(3n+3)!} \approx \lim_{n \rightarrow \infty} \frac{3n \ln(3n) - 3n}{(3n+3) \ln(3n+3) - 3n - 1} < 1$$

$$\lim_{n \rightarrow \infty} \frac{[(n+1)!]^3 e}{(n!)^3} > 1$$

$$\sum_{n=0}^{\infty} \frac{100^n}{n^{200}}$$

$$\rho_n = \frac{\frac{100^{n+1}}{(n+1)^{200}}}{\frac{100^n}{n^{200}}} = \frac{100^{n+1} n^{200}}{100^n (n+1)^{200}} = \frac{100 n^{200}}{(n+1)^{200}}$$

$$\rho = \lim_{n \rightarrow \infty} \frac{100n^{200}}{(n+1)^{200}} < 1 \therefore \text{the series converges}$$

$$\sum_{n=0}^{\infty} \frac{n! (2n)!}{(3n)!}$$

$$\rho_n = \frac{\frac{(n+1)! (2n+2)!}{(3n+3)!}}{\frac{n! (2n)!}{(3n)!}} = \frac{(3n)! (n+1)! (2n+2)!}{n! (2n)! (3n+3)!} = \frac{(n+1)(3n)! (2n+2)!}{(2n)! (3n+3)!}$$

$$\rho_2 = \frac{3 \cdot 6! \cdot 6!}{4! \cdot 9!} = \frac{3 \cdot 720}{2 \cdot 9 \cdot 8 \cdot 7} = \frac{2,160}{1,008}$$

$$\rho = \lim_{n \rightarrow \infty} \frac{(n+1)(3n)! (2n+2)!}{(2n)! (3n+3)!} > 1 \therefore \text{the series diverges}$$

$$\sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{n!}$$

$$\rho_n = \frac{\frac{\sqrt{(2n+2)!}}{(n+1)!}}{\frac{\sqrt{(2n)!}}{n!}} = \frac{n! \sqrt{(2n+2)!}}{(n+1)! \sqrt{(2n)!}} = \frac{\sqrt{(2n+2)!}}{(n+1) \sqrt{(2n)!}}$$

$$\rho_2 = \frac{\sqrt{6!}}{3\sqrt{4!}} = \frac{\sqrt{6 \cdot 5}}{3} = \frac{\sqrt{30}}{3} > 1$$

$$\rho = \lim_{n \rightarrow \infty} \frac{\sqrt{(2n+2)(2n+1)}}{n+1} > 1 \therefore \text{the series diverges}$$