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PROBLEMS, SECTION 2

$$\sum_{n=1}^{\infty}\frac{n}{2^n}=\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\cdots$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}=-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\cdots$$

$$\sum_{n=1}^{\infty}\frac{n}{n+5}=\frac{1}{6}+\frac{2}{7}+\frac{3}{8}+\frac{4}{9}+\cdots$$

$$\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n+1}=\frac{1}{2}+\frac{\sqrt{2}}{3}+\frac{\sqrt{3}}{4}+\frac{\sqrt{4}}{5}+\cdots$$

$$\sum_{n=1}^{\infty}\frac{2n(2n+1)}{3n+5}=\frac{2\cdot 3}{8}+\frac{4\cdot 5}{11}+\frac{6\cdot 7}{14}+\frac{8\cdot 9}{17}+\cdots=\frac{6}{8}+\frac{20}{11}+\frac{42}{14}+\frac{72}{17}+\cdots$$

$$\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}=\frac{1}{2!}+\frac{(2)^2}{4!}+\frac{(6)^2}{6!}+\frac{(24)^2}{8!}+\cdots=\frac{1}{2}+\frac{4}{24}+\frac{36}{720}+\frac{576}{40{,}320}+\cdots$$

$$\frac{1}{3}+\frac{2}{5}+\frac{4}{7}+\frac{8}{9}+\frac{16}{11}+\cdots=\sum_{n=0}^{\infty}\frac{2^n}{2n+3}$$

$$\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\cdots=\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}$$

$$\frac{1}{4}-\frac{1}{9}+\frac{1}{16}-\frac{1}{25}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n+1)^2}$$

$$\frac{1}{7}+\frac{2}{9}+\frac{3}{11}+\frac{4}{13}+\cdots=\sum_{n=1}^{\infty}\frac{n}{2n+5}$$

$$\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^{n+1}}$$

$$\frac{\ln 2}{2}-\frac{\ln 3}{3}+\frac{\ln 4}{4}-\frac{\ln 5}{5}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\ln(n+1)}{n+1}$$

PROBLEMS, SECTION 4

$$S=\sum_{n=1}^{\infty}\frac{1}{2^n}$$

$$|S-S_4|=\left|S-\sum_{n=1}^{4}\frac{1}{2^n}\right|=\left|1-\frac{1}{2}-\frac{1}{4}-\frac{1}{8}-\frac{1}{16}\right|=|1-0.9375|=0.0625$$

$$|S-S_5|=\left|1-0.9375-\frac{1}{32}\right|=|0.0625-0.03125|=0.03125$$

$$|S-S_6|=\left|0.03125-\frac{1}{64}\right|=0.016525$$

$$|S-S_7|=\left|0.01625-\frac{1}{128}\right|=0.0084375$$

$$S=\sum_{n=1}^{\infty}\frac{1}{5^n}<\sum_{n=1}^{\infty}\frac{1}{2^n}$$

$$|S-S_4|=\left|S-\sum_{n=1}^{4}\frac{1}{5^n}\right|=\left|0.25-\frac{1}{5}-\frac{1}{25}-\frac{1}{125}-\frac{1}{625}\right|=|0.25-0.2496|=0.0004$$

$$|S-S_5|=\left|0.0004-\frac{1}{3{,}125}\right|=0.00008$$

$$S=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots$$

$$\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$$

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}<\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$|S-S_4|=|1-0.883333\ldots|=0.1666\ldots$$

$$S=\sum_{n=1}^{\infty}\frac{1}{n!}=1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\cdots$$

$$\frac{1}{n!} < \frac{1}{2^n} \forall n \geq 3$$

$$S = \sum_{n=1}^{\infty} \frac{1}{n2^n} = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \cdots$$

$$\frac{1}{n2^n} < \frac{1}{2^n} \forall n \geq 2$$

$$S = \sum_{n=1}^{\infty} \frac{1}{2^n + 3^n} = \frac{1}{2+3} + \frac{1}{4+9} + \frac{1}{8+27} + \frac{1}{16+81} + \cdots = \frac{1}{5} + \frac{1}{13} + \frac{1}{35} + \frac{1}{97} + \cdots$$

$$2^n + 3^n > 2^n \forall n \geq 1$$

$$\therefore \frac{1}{2^n + 3^n} < \frac{1}{2^n} \forall n \geq 1$$

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots$$

$$T = \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots$$

$$\therefore \frac{1}{n^2} \geq \frac{1}{2^n}$$

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$$

$$\frac{1}{(n+1)^2} < \frac{1}{n(n+1)}$$

$$S = \sum_{n=2}^{\infty} \frac{1}{n^2 - 1} = \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \cdots$$

$$\frac{1}{n^2 - 1} = \frac{1}{(n+1)(n-1)} \forall n \geq 2$$

PROBLEMS, SECTION 5

$$\frac{1}{2} - \frac{4}{5} + \frac{9}{10} - \frac{16}{17} + \frac{25}{26} - \frac{36}{37} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2 + 1}$$

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}n^2}{n^2+1}\right| = \lim_{n\to\infty}\left(\frac{2n}{2n}\right) = 1 \therefore \textit{the series diverges}$$

$$\sqrt{2}+\frac{\sqrt{3}}{2}+\frac{\sqrt{4}}{3}+\frac{\sqrt{5}}{4}+\frac{\sqrt{6}}{5}+\cdots = \sum_{n=1}^{\infty}\frac{\sqrt{n+1}}{n}$$

$$\lim_{n\to\infty}\frac{\sqrt{n+1}}{n} = \lim_{n\to\infty}\frac{\frac{1}{2\sqrt{n+1}}}{1} = 0$$

$$\sum_{n=1}^{\infty}\frac{n+3}{n^2+10n}$$

$$\lim_{n\to\infty}\frac{n+3}{n^2+10n} = \lim_{n\to\infty}\frac{1}{2n+10} = 0$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n n^2}{(n+1)^2}$$

$$\lim_{n\to\infty}\left|\frac{(-1)^n n^2}{(n+1)^2}\right| = \frac{2n}{2n+2} = \frac{2}{2} = 1 \therefore \textit{the series diverges}$$

$$\sum_{n=1}^{\infty}\frac{n!}{n!+1}$$

$$\lim_{n\to\infty}\frac{n!}{n!+1} \approx \lim_{n\to\infty}\frac{n\ln n - n}{n\ln n - n + 1} = \frac{\ln n + 1 - 1}{\ln n + 1 - 1} = 1 \therefore \textit{the series diverges}$$

$$\sum_{n=1}^{\infty}\frac{n!}{(n+1)!}$$

$$\lim_{n\to\infty}\frac{n!}{(n+1)!} \approx \lim_{n\to\infty}\frac{n\ln n - n}{(n+1)\ln(n+1)-(n+1)} = \lim_{n\to\infty}\frac{\ln n + 1 - 1}{\ln(n+1)+1-1} = \lim_{n\to\infty}\frac{\ln n}{\ln(n+1)}$$

$$= \lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n\to\infty}\frac{n+1}{n} = 1 \therefore \textit{the series diverges}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n n}{\sqrt{n^3+1}}$$

$$\lim_{n\to\infty}\left|\frac{(-1)^n n}{\sqrt{n^3+1}}\right| = \lim_{n\to\infty} n(n^3+1)^{-1/2} = \lim_{n\to\infty}\left|(n^3+1)^{-1/2} - \frac{1}{2}\cdot 3n^3(n^3+1)^{-3/2}\right| > 0$$

$$\therefore \textit{the series diverges}$$

$$\sum_{n=1}^{\infty}\frac{\ln n}{n}$$

$$\lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{\frac{1}{n}}{1} = \lim_{n\to\infty}\frac{1}{n} = 0$$

$$\sum_{n=1}^{\infty}\frac{3^n}{2^n+3^n}$$

$$\lim_{n\to\infty}\frac{3^n}{2^n+3^n} = \lim_{n\to\infty}\frac{3^n\ln 3}{2^n\ln 2+3^n\ln 3} = 0$$

$$\sum_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right) = \sum_{n=2}^{\infty}\frac{n^2-1}{n^2}$$

$$\lim_{n\to\infty}\frac{n^2-1}{n^2} = \lim_{n\to\infty}\frac{2n}{2n} = 1 \therefore \textit{the series diverges}$$

PROBLEMS, SECTION 6

1. Show that

$$n! > 2^n \forall n \geq 4$$

$$4! = 24 > 2^4 = 16$$

$$5! = 120 > 2^5 = 32$$

$$6! = 720 > 2^6 = 64$$

2.

$$1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\frac{8}{2^4}+\cdots = 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$$

$$\lim_{n\to\infty}\frac{2^{n-1}}{2^n} = \frac{1}{2} > 0 \therefore \textit{the series diverges}$$

$$\frac{1}{n} < \frac{2^{n-1}}{2^n} = \frac{1}{2}\forall n \geq 3 \therefore \sum_{n=1}^{\infty}\frac{1}{n} \therefore \textit{the harmonic series diverges}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\cdots<1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$$

$$\frac{1}{n^2}<\frac{1}{2^{n-1}}\forall n\geq 2 \therefore \textit{the series diverges}$$

$$\sum_{n=1}^{\infty}\frac{1}{n2^n}<\sum_{n=1}^{\infty}\frac{1}{2^n}\therefore \textit{the series diverges}$$

$$\sum_{n=1}^{\infty}\frac{1}{2^n+3^n}<\sum_{n=1}^{\infty}\frac{1}{2^n}\therefore \textit{the series diverges}$$

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}<\sum_{n=1}^{\infty}\frac{1}{n}\therefore \textit{the series diverges}$$

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}=\lim_{n\to\infty}\frac{1}{\frac{1}{2\sqrt{n}}}>0\therefore \textit{the series diverges}$$

$$\sum_{n=1}^{\infty}\frac{1}{\ln n}<\sum_{n=1}^{\infty}\frac{1}{n}\therefore \textit{the series diverges}$$

$$\lim_{n\to\infty}\frac{1}{\ln n}=n\neq 0\therefore \textit{the series diverges}$$

There are 9 one-digit numbers (1-9), 90 two-digit numbers (10-99). How many three-digit numbers, four-digit numbers, etc. numbers are there?

$$\frac{9}{10}+\frac{90}{100}+\frac{900}{1000}+\frac{9000}{10{,}000}+\cdots=\frac{9}{10}+\frac{9}{10}+\frac{9}{10}+\frac{9}{10}+\cdots$$

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$$

$$\frac{1}{n}<\frac{9}{10}\forall n\geq 2,\therefore \textit{the harmonic series diverges}$$

PROBLEMS, SECTION 6 Continued

$$\sum_{n=2}^{\infty}\frac{1}{n\ln n}$$

$$\int\frac{dn}{n\ln n}<\int\frac{dn}{n}$$

$$\int \frac{dn}{n \ln n} = \int \frac{dn}{\ln n^n}$$

$$\sum_{n=1}^{\infty} \frac{n}{n^2+4}$$

$$\lim_{n\to\infty} \frac{n}{n^2+4} = \lim_{n\to\infty} \frac{1}{2n} = 0$$

$$\int \frac{ndn}{n^2+4} = \frac{1}{2}\int d\ln(n^2+4) = \frac{1}{2}\ln(n^2+4) + C \therefore \textit{the series diverges}$$

$$\sum_{n=3}^{\infty} \frac{1}{n^2-4}$$

$$\frac{1}{n^2-4} = \frac{1}{(n-2)(n+2)} = \frac{A}{n-2} + \frac{B}{n+2}$$

$$\int \frac{dn}{n^2-4} = \int \frac{dn}{(n-2)(n+2)} = \int \frac{Adn}{n-2} + \int \frac{Bdn}{n+2} = A\ln(n-2) + B\ln(n+2) + C \therefore \textit{the series diverges}$$

$$\sum_{n=1}^{\infty} \frac{e^n}{e^{2n}+9}$$

$$\int \frac{e^n dn}{e^{2n}+9} = \int \frac{e^n dn}{(e^n-3)(e^n+3)} = \int \frac{Ae^n dn}{e^n-3} + \int \frac{Be^n dn}{e^n+3} = A\ln(e^n-3) + B\ln(e^n+3) + C \therefore \textit{the series diverges}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(1+\ln n)^{3/2}}$$

$$\int \frac{dn}{n(1+\ln n)^{3/2}} = \int d(1+\ln n)^{1/2} = (1+\ln n)^{1/2} + C \rightarrow \textit{the series diverges}$$

$$\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$$

$$\int \frac{ndn}{(n^2+1)^2} = \int n\,(n^2+1)^{-1}dn = -\frac{1}{2}\int d\,(n^2+1)^{-1}dn = -\frac{1}{2}(n^2+1)^{-1} + C$$

$$\rightarrow \textit{the series converges}$$

$$\sum_{n=}^{\infty} \frac{n^2}{n^3+1}$$

$$\int \frac{n^2 dn}{n^3+1} = \frac{1}{3}\ln(n^3+1) + C \rightarrow \infty \therefore \textit{the series diverges}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+9}}$$

$$\int \frac{dn}{\sqrt{n^2+9}} = \int \frac{dn}{\sqrt{(n-3)(n+3)}}$$

$$\frac{1}{\sqrt{n^2+9}} = \frac{1}{\sqrt{9\left[\left(\frac{n}{3}\right)^2+1\right]}} = \frac{1}{3\sqrt{\left(\frac{n}{3}\right)^2+1}}$$

$$\int \frac{dn}{\sqrt{n^2+9}} = \frac{1}{3}\int \left[\left(\frac{n}{3}\right)^2+1\right]^{1/2} dn = \int (m^2+1)^{-1/2}dm$$

$$m = \tan x$$

$$(\cos x)^2 + (\sin x)^2 = 1$$

$$(\cos x)^2 = 1 - (\sin x)^2$$

$$1 = \frac{1}{(\cos x)^2} - (\tan x)^2 = (\sec x)^2 - (\tan x)^2$$

$$(\sec x)^2 = 1 + (\tan x)^2$$

$$dm = d\tan x = d\left(\frac{\sin x}{\cos x}\right) = \left[\frac{\cos x}{\cos x} + (\tan x)^2\right]dx = [1+(\tan x)^2]dx = (\sec x)^2 dx$$

$$\int (m^2+1)^{-1/2}dm = \int \sec x\,dx = \int \frac{dx}{\cos x} = \tan x \sec x + C$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

Suppose p = 1 then we have the divergent harmonic series by the integral test:

$$\int \frac{dn}{n} = \ln n + C$$

$$\int \frac{dn}{n^p} = \int n^{-p}\,dn = -\frac{n^{-p+1}}{(1-p)} + C \therefore converges\ for\ p > 1, otherwise\ diverges$$

$$\sum_{n=1}^{\infty} e^{-n^2}$$

$$\int \frac{dn}{e^{n^2}} < \int \frac{dn}{e^n} = -e^{-n}\ln e \to 0\ as\ n \to \infty$$

PROBLEMS, SECTION 6 Continued

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2}$$

$$\rho_n = \frac{\frac{2^{n+1}}{(n+1)^2}}{\frac{2^n}{n^2}} = \frac{n^2 2^{n+1}}{2^n (n+1)^2} = \frac{2n^2}{(n+1)^2}$$

$$\rho = \lim_{n\to\infty} \frac{2n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{4n}{2n+1} = \frac{4}{2} = 2 \therefore the\ series\ diverges$$

$$\sum_{n=1}^{\infty} \frac{3^n}{2^{2n}}$$

$$\rho_n = \frac{\frac{3^{n+1}}{2^{2n+2}}}{\frac{3^n}{2^{2n}}} = \frac{2^{2n} 3^{n+1}}{2^{2n+2} 3^n} = \frac{3}{2^2} = \frac{3}{4}$$

$$\rho = \lim_{n\to\infty} \frac{3}{4} = \frac{3}{4} < 1 \therefore the\ series\ converges$$

$$\sum_{n=0}^{\infty} \frac{n!}{(2n)!}$$

$$\rho_n = \frac{\frac{(n+1)!}{[2(n+1)]!}}{\frac{n!}{(2n)!}} = \frac{(2n)!\,(n+1)!}{n!\,[2(n+1)]!}$$

$$\rho_2 = \frac{4!\,3!}{2!\,6!} = \frac{4 \cdot 3 \cdot 6}{720} = \frac{12 \cdot 6}{720} = \frac{72}{720} < 1$$

$$\rho = \lim_{n\to\infty} \frac{(2n)!\,(n+1)!}{n!\,[2(n+1)]!} < 1 \therefore \textit{the series converges}$$

$$\sum_{n=0}^{\infty} \frac{5^n (n!)^2}{(2n)!}$$

$$\rho_n = \frac{\frac{5^{n+1}[(2n+2)!]^2}{[2(n+1)]!}}{\frac{5^n (n!)^2}{(2n)!}} = \frac{(2n)!\,5^{n+1}[(2n+2)!]^2}{5^n (n!)^2 [2(n+1)]!} = \frac{(2n)!\,[(2n+2)!]^2}{(n!)^2[2(n+1)]!}$$

$$\rho > 1 \therefore \textit{the series diverges}$$

$$\sum_{n=1}^{\infty} \frac{10^n}{(n!)^2}$$

$$\rho_n = \frac{\frac{10^{n+1}}{[(n+1)!]^2}}{\frac{10^n}{(n!)^2}} = \frac{10^{n+1}(n!)^2}{10^n[(n+1)!]^2} = \frac{10(n!)^2}{[(n+1)!]^2}$$

$$\rho_2 = \frac{10 \cdot 4}{36} = \frac{40}{36}$$

$$\rho_3 = \frac{10 \cdot 36}{576} = \frac{360}{576}$$

$$\rho = \lim_{n\to\infty} \frac{10(n!)^2}{[(n+1)!]^2} < 1 \therefore \textit{the series converges}$$

$$\sum_{n=1}^{\infty} \frac{n!}{100^n}$$

$$\rho_n = \frac{\frac{(n+1)!}{100^{n+1}}}{\frac{n!}{100^n}} = \frac{100^n (n+1)!}{100^{n+1} n!} \approx \frac{(n+1)\ln(n+1) - (n+1)}{100(n \ln n - n)}$$

$$\rho = \lim_{n\to\infty} \frac{(n+1)\ln(n+1) - (n+1)}{100(n \ln n - n)} > 1 \therefore \textit{the series diverges}$$

$$\sum_{n=0}^{\infty} \frac{3^{2n}}{2^{3n}}$$

$$\rho_n = \frac{\frac{3^{2n+2}}{2^{3n+3}}}{\frac{3^{2n}}{2^{3n}}} = \frac{3^{2n+2}2^{3n}}{3^{2n}2^{3n+3}} = \frac{9}{8}$$

$$\rho = \frac{9}{8} \therefore \textit{the series diverges}$$

$$\sum_{n=1}^{\infty} \frac{e^n}{\sqrt{n!}}$$

$$\rho_n = \frac{\frac{e^{n+1}}{\sqrt{(n+1)!}}}{\frac{e^n}{\sqrt{n!}}} = \frac{e\sqrt{n!}}{\sqrt{(n+1)!}}$$

$$\rho = \lim_{n\to\infty} \frac{e\sqrt{n!}}{\sqrt{(n+1)!}} < 1 \therefore \textit{the series converges}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^3 e^{3n}}{(3n)!}$$

$$\rho_n = \frac{\frac{[(n+1)!]^3 e^{3n+3}}{(3n+3)!}}{\frac{(n!)^3 e^{3n}}{(3n)!}} = \frac{(3n)!\,[(n+1)!]^3 e^{3n+3}}{(n!)^3 (3n+3)!\, e^{3n}}$$

$$\rho = \lim_{n\to\infty} \frac{(3n)!\,[(n+1)!]^3 e^3}{(n!)^3 (3n+3)!}$$

$$\lim_{n\to\infty} \frac{(3n)!}{(3n+3)!} \approx \lim_{n\to\infty} \frac{3n\ln(3n) - 3n}{(3n+3)\ln(3n+3) - 3n - 1} < 1$$

$$\lim_{n\to\infty} \frac{[(n+1)!]^3 e}{(n!)^3} > 1$$

$$\sum_{n=0}^{\infty} \frac{100^n}{n^{200}}$$

$$\rho_n = \frac{\frac{100^{n+1}}{(n+1)^{200}}}{\frac{100^n}{n^{200}}} = \frac{100^{n+1} n^{200}}{100^n (n+1)^{200}} = \frac{100 n^{200}}{(n+1)^{200}}$$

$$\rho = \lim_{n\to\infty} \frac{100n^{200}}{(n+1)^{200}} < 1 \therefore \textit{the series converges}$$

$$\sum_{n=0}^{\infty} \frac{n!\,(2n)!}{(3n)!}$$

$$\rho_n = \frac{\frac{(n+1)!\,(2n+2)!}{(3n+3)!}}{\frac{n!\,(2n)!}{(3n)!}} = \frac{(3n)!\,(n+1)!\,(2n+2)!}{n!\,(2n)!\,(3n+3)!} = \frac{(n+1)(3n)!\,(2n+2)!}{(2n)!\,(3n+3)!}$$

$$\rho_2 = \frac{3 \cdot 6! \cdot 6!}{4! \cdot 9!} = \frac{3 \cdot 720}{2 \cdot 9 \cdot 8 \cdot 7} = \frac{2{,}160}{1{,}008}$$

$$\rho = \lim_{n\to\infty} \frac{(n+1)(3n)!\,(2n+2)!}{(2n)!\,(3n+3)!} > 1 \therefore \textit{the series diverges}$$

$$\sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{n!}$$

$$\rho_n = \frac{\frac{\sqrt{(2n+2)!}}{(n+1)!}}{\frac{\sqrt{(2n)!}}{n!}} = \frac{n!\sqrt{(2n+2)!}}{(n+1)!\sqrt{(2n)!}} = \frac{\sqrt{(2n+2)!}}{(n+1)\sqrt{(2n)!}}$$

$$\rho_2 = \frac{\sqrt{6!}}{3\sqrt{4!}} = \frac{\sqrt{6 \cdot 5}}{3} = \frac{\sqrt{30}}{3} > 1$$

$$\rho_3 = \frac{\sqrt{8!\,7!}}{4} > 1$$

$$\rho = \lim_{n\to\infty} \frac{\sqrt{(2n+2)(2n+1)}}{n+1} > 1 \therefore \textit{the series diverges}$$

PROBLEMS, SECTION 6 Continued

$$\sum_{n=9}^{\infty} \frac{(2n+1)(3n-5)}{\sqrt{n^2-73}}$$

$$\lim_{n\to\infty} \frac{\frac{[2(n+1)+1][3(n+1)-5]}{n^2-73}}{\frac{(2n+1)(3n-5)}{\sqrt{n^2-73}}} = \lim \frac{\sqrt{n^2-73}[2(n+1)+1][3(n+1)-5]}{(n^2-73)\underset{n\to\infty}{(2n+1)}(3n-5)} = \infty$$

$$\therefore \textit{the series diverges by the rato test}$$

$$\sum_{n=0}^{\infty}\frac{n(n+1)}{(n+2)^2(n+3)}=\sum_{n=0}^{\infty}\frac{n^2+n}{(n^2+4n+4)(n+3)}$$

$$\sum_{n=0}^{\infty}\frac{n^2+n}{(n^2+4n+4)(n+3)}=\sum_{n=0}^{\infty}\frac{n^2+n}{n^3+4n^2+4n+3n^2+12n+12}$$
$$=\sum_{n=0}^{\infty}\frac{n^2+n}{n^3+7n^2+16n+12}$$

$$\lim_{n\to\infty}\frac{n^2+n}{n^3+7n^2+16n+12}=\lim_{n\to\infty}\frac{2n+1}{3n^2+14n+16}=\lim_{n\to\infty}\frac{2}{6n+14}=0$$

$$\sum_{n=6}^{\infty}\frac{1}{2^n-n^2}<\sum_{n=6}^{\infty}\frac{1}{n^2}\therefore \textit{the series converges}$$

$$\sum_{n=1}^{\infty}\frac{n^2+3n+4}{n^4+7n^3+6n-1}$$

$$\lim_{n\to\infty}\frac{n^2+3n+4}{n^4+7n^3+6n-1}=\lim_{n\to\infty}\frac{2n+3}{4n^3+14n^2+6}=\lim_{n\to\infty}\frac{2}{12n^2+28n}=0$$

$$\sum_{n=1}^{\infty}\frac{n^2+3n+4}{n^4+7n^3+6n-1}<\sum_{n=1}^{\infty}\frac{1}{n^2}\therefore \textit{the series converges}$$

$$\sum_{n=3}^{\infty}\frac{(n-\ln n)^2}{5n^4-3n^2+1}$$

$$\lim_{n\to\infty}\frac{(n-\ln n)^2}{5n^4-3n^2+1}=\lim_{n\to\infty}\frac{n^2-2\ln n+(\ln n)^2}{20n^3-6n}=\lim_{n\to\infty}\frac{2n-2+2\ln n}{60n^2-6}=\lim_{n\to\infty}\frac{2+\frac{2}{n}}{120n-6}=0$$

$$\sum_{n=30}^{\infty}\frac{(n-\ln n)^2}{5n^4-3n^2+1}<\sum_{n=30}^{\infty}\frac{1}{n^2}\therefore \textit{the series converges}$$

$$\sum_{n=30}^{\infty}\frac{\sqrt{n^3+5n-1}}{n^2-\sin n^3}<\sum_{n=30}^{\infty}\frac{1}{n^2}\therefore \textit{the series converges}$$