PROBLEMS, SECTION 7

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} (1.)$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = 1(?)$$

$$\frac{1}{\sqrt{2}}, \frac{\sqrt{2}}{\sqrt{3}}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{\sqrt{5}}, \frac{\sqrt{5}}{\sqrt{6}}, \dots = 0.707107, 0.816497, 0.866025, \dots$$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

$$\sqrt{3} > \sqrt{2}, \sqrt{4} < \sqrt{3}$$

$$\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} \therefore \textit{the series converges}$$

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2} (2.)$$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{2^n}{n^2} = \infty \therefore \textit{the series diverges}$$

$$2, \frac{4}{4}, \frac{8}{9}, \frac{16}{16}, \frac{32}{25}, \frac{64}{36}, \dots$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (3.)$$

$$\lim_{n\to\infty} |a_n| = 0$$

$$\frac{1}{(n+1)^2} < \frac{1}{n^2} \therefore \textit{the series converges}$$

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n!} (4.)$$

$$\frac{3}{1}, \frac{9}{2}, \frac{27}{6}, \frac{81}{24}, \frac{243}{120}, \dots$$

$$\lim_{n\to\infty} |a_n| \neq 0 \therefore \textit{the series diverges}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n} (5.)$$

$$\lim_{n\to\infty} |a_n| = 0$$

$$\lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

$$\ln 3 > \ln 2, etc. \therefore \textit{the series diverges}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+5} (6.)$$

$$\lim_{n\to\infty} \frac{n}{n+5} = 1 \therefore \textit{the series diverges}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n n}{1+n^2} (7.)$$

$$\lim_{n\to\infty} \frac{n}{1+n^2} = 0$$

$$\frac{1}{2}, \frac{2}{5}, \dots$$

$$\lim_{n\to\infty} \frac{\frac{(1+n)}{1+(1+n^2)^2}}{\frac{n}{1+n^2}} = \lim_{n\to\infty} \frac{(1+n^2)(1+n)}{n[1+(1+n^2)^2]} = 0 \therefore \textit{the series converges}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{10n}}{n+2} (8.)$$

$$\frac{\sqrt{10}}{3}, \frac{\sqrt{20}}{4}, \frac{\sqrt{30}}{5} \therefore \textit{the series diverges}$$

PROBLEMS, SECTION 9

$$\sum_{n=1}^{\infty} \frac{n-1}{(n+2)(n+3)}$$

$$\lim_{n\to\infty} \frac{n-1}{(n+2)(n+3)} = \lim_{n\to\infty} \frac{n-1}{n^2+5n+6} = \lim_{n\to\infty} \frac{1}{2n+5} = 0$$

$$\frac{n-1}{(n+2)(n+3)} = \frac{a}{n+2} + \frac{b}{n+3}$$

$$a(n+3) + b(n+2) = n-1$$

$$(a+b)n + 3a + 2b = n-1$$

$$a+b = 1, 3a+2b = -1$$

$$3a + 2b = -1$$
$$2a + 2b = 2$$

$$a = -3, b = 4, -9 + 8 = -1$$

$$\int \frac{n-1}{(n+2)(n+3)} dn = \int \frac{-3}{n+2} dn + \int \frac{4}{n+3} dn = -3\ln(n+2) + 4\ln(n+3) + C$$

The series diverges by the integral test.

$$\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1}$$

$$\lim_{n\to\infty} \frac{n^2-1}{n^2+1} = \lim_{n\to\infty} \frac{2n}{2n} = 1 \therefore \textit{the series diverges}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{\ln 3}}$$

$$\ln 3 = 1.0986122886681096913952452369225 > 1 \therefore \textit{the series converges}$$

$$\sum_{n=0}^{\infty} \frac{n^2}{n^3+4}$$

$$\lim_{n\to\infty} \frac{n^2}{n^3+4} = \lim_{n\to\infty} \frac{2n}{3n^2} = \lim_{n\to\infty} \frac{2}{6n} = 0$$

$$\lim_{n\to\infty}\frac{\frac{(n+1)^2}{(n+1)^3+4}}{\frac{n^2}{n^3+4}}=\lim_{n\to\infty}\frac{(n^3+4)(n+1)^2}{n^2[(n+1)^3+4]}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$=\lim_{n\to\infty}\frac{(n^3+4)(n^2+2n+1)}{n^2(n^3+3n^2+3n+1+4)}=\lim_{n\to\infty}\frac{n^5+2n^4+n^3+4n^2+8n+4}{n^5+3n^4+3n^3+5n^2}$$

$$=\lim_{n\to\infty}\frac{5n^4+8n^3+3n^2+8n}{5n^4+12n^3+9n^2+10n}$$

$$=\lim_{n\to\infty}\frac{20n^3+24n^2+6n+8}{20n^3+36n^2+18n+10}$$

$$=\lim_{n\to\infty}\frac{60n^2+48n+6}{60n^2+72n+18}=\lim_{n\to\infty}\frac{120n+48}{120n+72}=\lim_{n\to\infty}\frac{120}{120}=1$$

$$\frac{d}{dn}\ln(n^3+4)=\frac{3n^2}{n^3+4}$$

$$\int\frac{n^2}{n^3+4}dn=\frac{1}{3}\ln(n^3+4)+C\to\infty\ as\ n\to\infty\ \therefore\ the\ series\ diverges$$

$$\sum_{n=1}^{\infty}\frac{n}{n^3-4}$$

$$\lim_{n\to\infty}\frac{n}{n^3-4}=\lim_{n\to\infty}\frac{1}{3n^2}=0$$

$$\frac{1}{-3},\frac{2}{4},\frac{3}{23},\frac{4}{60},\frac{5}{121},\frac{6}{216}$$

$$\lim_{n\to\infty}\frac{\frac{n+1}{(n+1)^3-4}}{\frac{n}{n^3-4}}=\lim_{n\to\infty}\frac{(n^3-4)(n+1)}{n[(n+1)^3-4]}\neq 0\ \therefore\ the\ series\ diverges$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}$$

$$\lim_{n\to\infty}\frac{\frac{[(n+1)!]^2}{[2(n+1)]!}}{\frac{(n!)^2}{(2n)!}}=\lim_{n\to\infty}\frac{(2n)!\,[(n+1)!]^2}{(n!)^2[2(n+1)]!}<1\ \therefore\ the\ series\ converges\ see\ table\ as\ follows$$

n	Ratio
0	0.500000000000000
1	0.333333333333333
2	0.300000000000000
3	0.285714285714286
4	0.277777777777778
5	0.272727272727273
6	0.269230769230769
7	0.266666666666667
8	0.264705882352941
9	0.263157894736842
10	0.261904761904762
11	0.260869565217391
12	0.260000000000000
13	0.259259259259259
14	0.258620689655172
15	0.258064516129032
16	0.257575757575758
17	0.257142857142857
18	0.256756756756757
19	0.256410256410256
20	0.256097560975610
21	0.255813953488372
22	0.255555555555556
23	0.255319148936170
24	0.255102040816327
25	0.254901960784314

$$\sum_{n=0}^{\infty} \frac{(2n)!}{3^n (n!)^2}$$

$$\lim_{n\to\infty} \frac{\dfrac{(2n+2)!}{3^{n+1}[(n+1)!]^2}}{\dfrac{(2n)!}{3^n(n!)^2}} = \lim_{n\to\infty} \frac{3^n(n!)^2(2n+2)!}{3^{n+1}[(n+1)!]^2(2n)!} > 1$$

$$\therefore \textit{the series diverges see table as follows}$$

n	Ratio
0	1.000000000000000
1	0.666666666666667
2	0.666666666666667
3	0.740740740740741
4	0.864197530864197
5	1.037037037037037
6	1.267489711934156
7	1.569272976680384
8	1.961591220850480
9	2.470151907737641
10	3.128859083134346
11	3.982184287625532
12	5.088346589743734

$$\sum_{n=1}^{\infty} \frac{n^5}{5^n}$$

$$\lim_{n\to\infty} \frac{\frac{(n+1)^5}{5^{n+1}}}{\frac{n^5}{5^n}} = \lim_{n\to\infty} \frac{5^n (n+1)^5}{n^5 5^{n+1}} =$$

$$= \lim_{n\to\infty} \frac{n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1}{5n^5}$$

$$= \lim_{n\to\infty} \frac{5n^4 + 20n^3 + 30n^2 + 20n + 5}{20n^4}$$

$$= \lim_{n\to\infty} \frac{20n^3 + 60n^2 + 60n + 20}{80n^3} = \lim_{n\to\infty} \frac{60n^2 + 120n + 60}{240n^2}$$

$$= \lim_{n\to\infty} \frac{120n + 120}{480n} = \lim_{n\to\infty} \frac{120}{480} = \frac{1}{4} < 1$$

$$\therefore \textit{the series converges, also see table as follows}$$

n	Ratio
1	6.400000000000000
2	1.518750000000000
3	0.842798353909465
4	0.610351562500000
5	0.497664000000000
6	0.432278806584362
7	0.389932766109359
8	0.360406494140625
9	0.338701756168606
10	0.322102000000000
11	0.309010189318911
12	0.298428658693416

13	**0.289703280158796**
14	**0.282387918724341**
15	**0.276168164609054**

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

$$\lim_{n\to\infty} \frac{\frac{n^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \lim_{n\to\infty} \frac{n^{n+1}n!}{n^n(n+1)!} = \lim_{n\to\infty} \frac{nn!}{(n+1)!} = \lim_{n\to\infty} \frac{n}{n+1} = 1 \therefore \textit{we need another test}$$

n	**Term**
1	**1**
2	**2**
3	**4**
4	**11**
5	**26**
6	**65**
7	**163**
8	**416**
9	**1068**
10	**2756**
11	**7148**
12	**18614**
13	**48639**
14	**127463**
15	**334865**

The terms increase without a bound or limit therefore the series diverges.

$$\sum_{n=2}^{\infty} \frac{(-1)^n n}{n-1}$$

$$\lim_{n\to\infty} \frac{n}{n-1} = 1 \therefore \textit{the series diverges}$$

$$\lim_{n\to\infty} \frac{\frac{n+1}{n}}{\frac{n}{n-1}} = \lim_{n\to\infty} \frac{(n-1)(n+1)}{n^2}$$

$$= \lim_{n\to\infty} \frac{n^2-1}{n^2} = \lim_{n\to\infty} \frac{2n}{2n} = 1 \therefore \textit{we needed the previous test}$$

$$\sum_{n=4}^{\infty} \frac{2n}{n^2-9}$$

$$\lim_{n\to\infty} \frac{2n}{n^2-9} = \lim_{n\to\infty} \frac{2}{2n} = 0$$

$$\lim_{n\to\infty} \frac{\frac{2(n+1)}{(n+1)^2-9}}{\frac{2n}{n^2-9}} = 1\ (?), \mathit{see\ table\ below, we\ need\ another\ test}$$

n	Term
4	0.546875000000
5	0.711111111111
6	0.787500000000
7	0.831168831169
8	0.859375000000
9	0.879120879121
10	0.893750000000
11	0.905050505051
12	0.914062500000
13	0.921431509667
14	0.927579365079
15	0.932793522267
16	0.937276785714
17	0.941176470588
18	0.944602272727
19	0.947637636290
20	0.950347222222
21	0.952781954887
22	0.954982517483
23	0.956981826547
24	0.958806818182
25	0.960479760120

$$\frac{2n}{n^2-9} = \frac{2n}{(n-3)(n+3)} = \frac{A}{n-3} + \frac{B}{n+3}$$

$$A(n+3) + B(n-3) = 2n$$

$$6A = 6 \therefore A = 1$$

$$B(-3-3) = -6 \therefore B = 1$$

$$\int \left(\frac{1}{n-3} + \frac{1}{n+3}\right) dn = \ln(n-3) + \ln(n+3)$$

The previous series diverges by the integral test.

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - n}$$

$$\lim_{n\to\infty} \frac{1}{n^2 - n} = 0$$

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)^2 - n - 1}}{\frac{1}{n^2 - n}} = \lim_{n\to\infty} \frac{n^2 - n}{[(n+1)^2 - n - 1]} = \lim_{n\to\infty} \frac{2n - 1}{2(n+1) - 1} = \lim_{n\to\infty} \frac{2}{2} = 1$$

$$\therefore \textit{we need another test}$$

$$\frac{1}{n^2 - n} = \frac{1}{n(n-1)} = \frac{A}{n} + \frac{B}{n-1} \therefore \textit{the series diverges by the integral tests}$$

$$\int \frac{A}{n} dn = \mathrm{A} \ln n + C \textit{ and } \int \frac{B}{n-1} dn = \ln(n-1) + D$$