Blog Entry © Thursday, July 31, 2025, Numerically Solving Four Second Order Linear Ordinary Differential Equation Boundary Value Problems by James Pate Williams, Jr.

The first equation is:

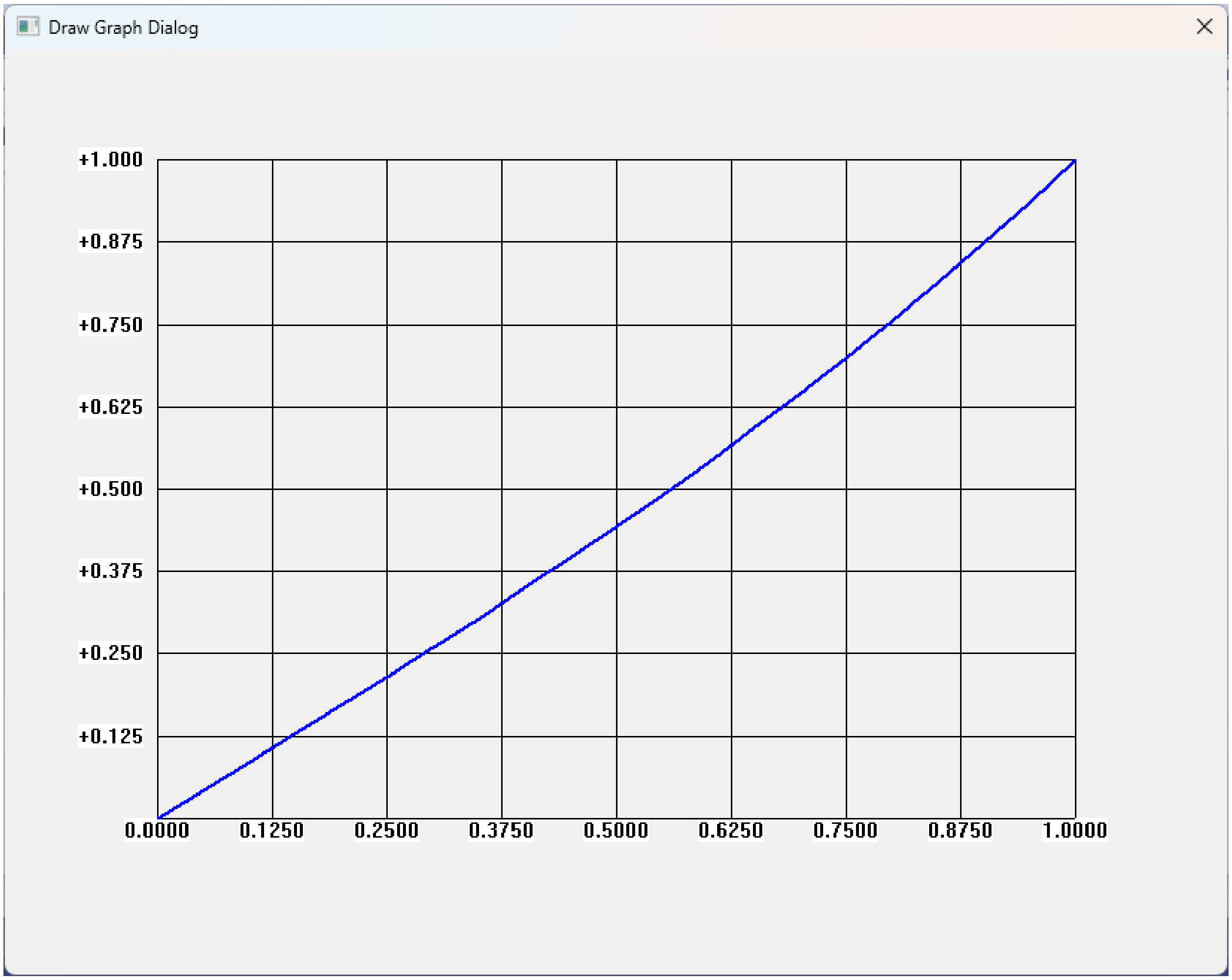
$$y''(x) - y(x) = 0, y(0) = 0, y(1) = 1$$

The solutions are found by using finite differences. See ***Elementary Numerical Analysis: An Algorithmic Approach Third Edition*** © 1980 by S. D. Conte and Carl de Boor Chapter Nine.

The exact solution is given by the equation:

$$y(x) = \frac{\sinh x}{\sinh 1}$$

n	x	y	exact	% error
0	0.00	0.0000000000	0.0000000000	0.0000000000
1	0.05	0.0425650199	0.0425636361	0.0032510945
2	0.10	0.0852364523	0.0852337034	0.0032250823
3	0.15	0.1281209758	0.1281168994	0.0031817864
4	0.20	0.1713258018	0.1713204544	0.0031212927
5	0.25	0.2149589423	0.2149523998	0.0030437211
6	0.30	0.2591294802	0.2591218381	0.0029492237
7	0.35	0.3039478417	0.3039392160	0.0028379848
8	0.40	0.3495260729	0.3495166002	0.0027102186
9	0.45	0.3959781192	0.3959679580	0.0025661688
10	0.50	0.4434201109	0.4434094420	0.0024061059
11	0.55	0.4919706528	0.4919596805	0.0022303262
12	0.60	0.5417511213	0.5417400745	0.0020391497
13	0.65	0.5928859677	0.5928751008	0.0018329179
14	0.70	0.6455030290	0.6454926237	0.0016119920
15	0.75	0.6997338478	0.6997242144	0.0013767504
16	0.80	0.7557140013	0.7557054800	0.0011275869
17	0.85	0.8135834397	0.8135764031	0.0008649083
18	0.90	0.8734868368	0.8734816908	0.0005891322
19	0.95	0.9355739510	0.9355711378	0.0003006851
20	1.00	1.0000000000	1.0000000000	0.0000000000



Another equation from the same textbook is:

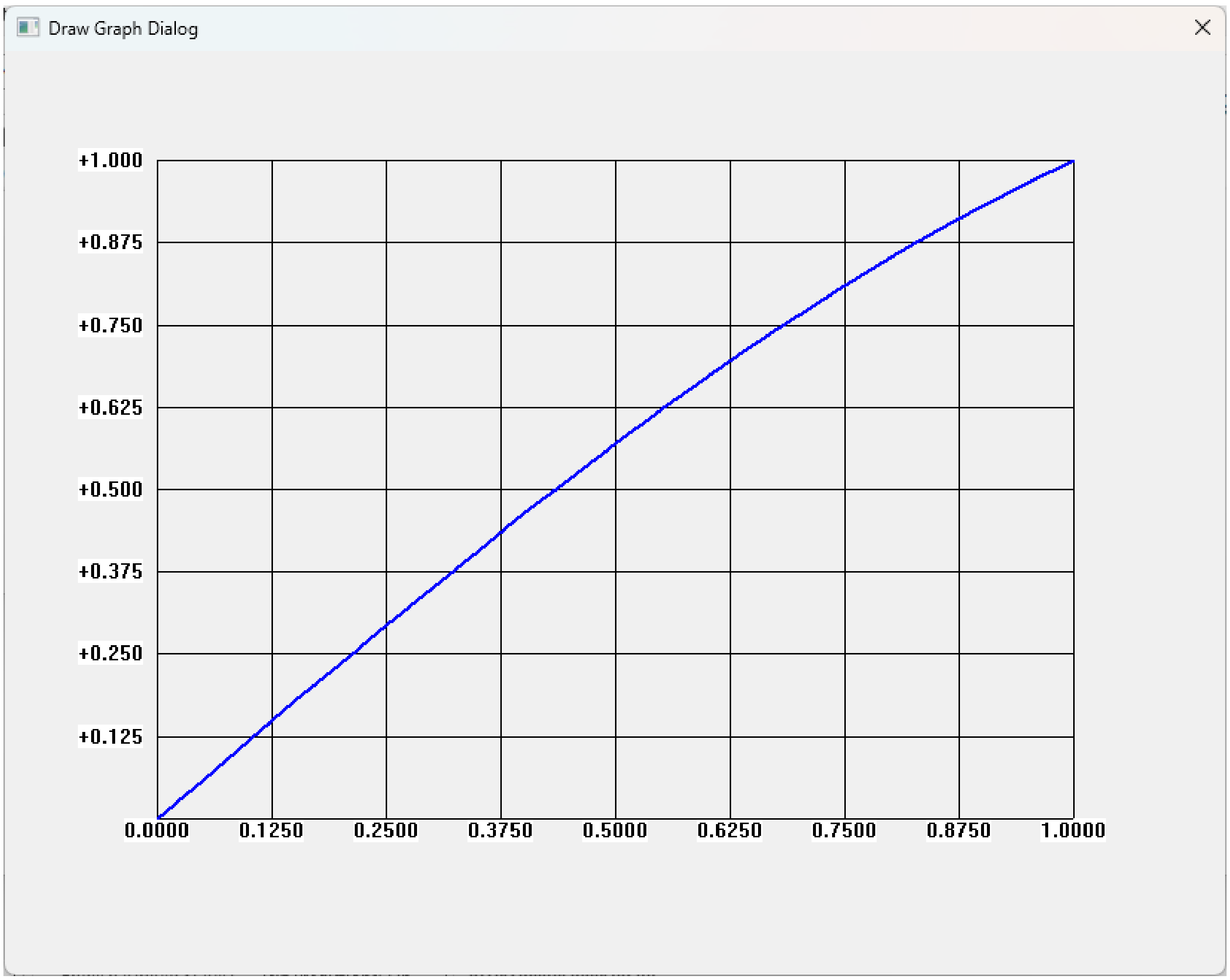
$$y''(x) + y(x) = 0, y(0) = 0, y(1) = 1$$

The exact solution is:

$$y(x) = \frac{\sin x}{\sin 1}$$

n	x	y	exact	% error
0	0.00	0.0000000000	0.0000000000	0.0000000000
1	0.05	0.0593972102	0.0593950002	0.0037208590
2	0.10	0.1186459273	0.1186415437	0.0036947859
3	0.15	0.1775980296	0.1775915452	0.0036512728
4	0.20	0.2361061368	0.2360976604	0.0035902320
5	0.25	0.2940239787	0.2940136543	0.0035115400
6	0.30	0.3512067607	0.3511947673	0.0034150367
7	0.35	0.4075115257	0.4074980762	0.0033005238
8	0.40	0.4627975120	0.4627828521	0.0031677637

9	0.45	0.5169265044	0.5169109119	0.0030164774
10	0.50	0.5697631806	0.5697469637	0.0028463427
11	0.55	0.6211754489	0.6211589447	0.0026569916
12	0.60	0.6710347785	0.6710183519	0.0024480071
13	0.65	0.7192165211	0.7192005627	0.0022189202
14	0.70	0.7656002225	0.7655851466	0.0019692059
15	0.75	0.8100699233	0.8100561663	0.0016982786
16	0.80	0.8525144493	0.8525024675	0.0014054870
17	0.85	0.8928276892	0.8928179565	0.0010901083
18	0.90	0.9309088599	0.9309018656	0.0007513415
19	0.95	0.9666627584	0.9666590049	0.0003882994
20	1.00	1.0000000000	1.0000000000	0.0000000000
20	1.00	1.0000000000	1.0000000000	0.0000000000



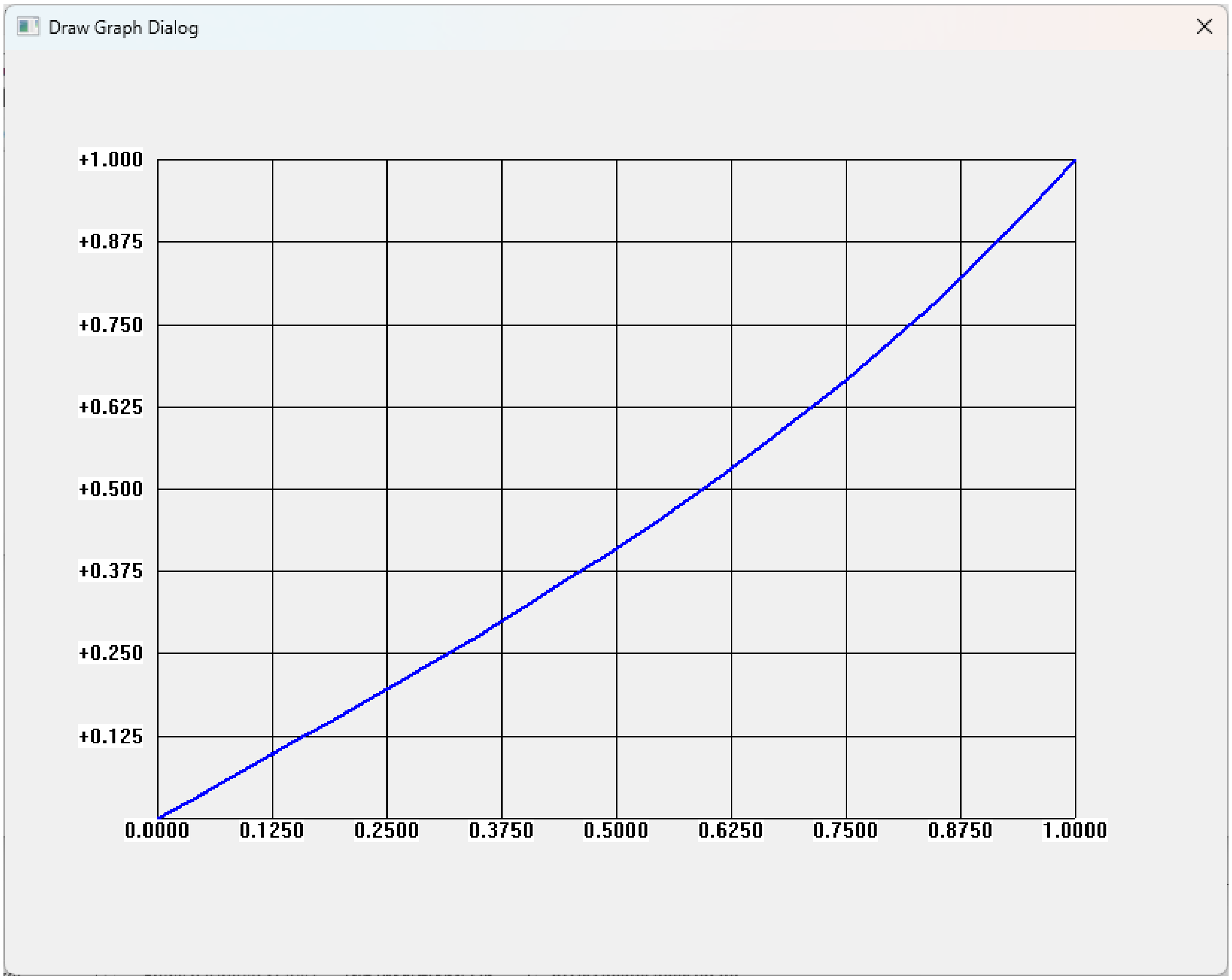
The third equation is a modification of the first equation:

$$y''(x) - y(x) = x^2, y(0) = 0, y(1) = 1$$

According to the Wolfram website the exact solution is:

$$y(x) = \frac{\left[e^{-x}\left\{e^{x}(x^2+2) - e^{(x+2)}(x^2+2\right\} - 2e^{2x} + 4e^{2x+1} + 2e^2 - 4e\right]}{e^2 - 1}$$

n	x	y	exact	% error
0	0.00	0.0000000000	0.0000000000	0.0000000000
1	0.05	0.0389079814	0.0388968200	0.0286948465
2	0.10	0.0779194828	0.0778981954	0.0273271451
3	0.15	0.1171507829	0.1171204039	0.0259382677
4	0.20	0.1567312099	0.1566927778	0.0245270550
5	0.25	0.1968034650	0.1967580279	0.0230928915
6	0.30	0.2375239787	0.2374725998	0.0216357213
7	0.35	0.2790633024	0.2790070656	0.0201560568
8	0.40	0.3216065343	0.3215465499	0.0186549781
9	0.45	0.3653537826	0.3652911932	0.0171341211
10	0.50	0.4105206653	0.4104566519	0.0155956550
11	0.55	0.4573388497	0.4572746381	0.0140422471
12	0.60	0.5060566312	0.5059934983	0.0124770166
13	0.65	0.5569395543	0.5568788351	0.0109034766
14	0.70	0.6102710763	0.6102141710	0.0093254673
15	0.75	0.6663532759	0.6663016570	0.0077470807
16	0.80	0.7255076088	0.7254628290	0.0061725795
17	0.85	0.7880757107	0.7880394111	0.0046063142
18	0.90	0.8544202518	0.8543941703	0.0030526383
19	0.95	0.9249258436	0.9249118235	0.0015158268
20	1.00	1.0000000000	1.0000000000	0.0000000000



Last, we partially solved a Euler-Cauchy equation using a Frobenius infinite power series.

$$x^2y''(x) + xy(x) + y(x) = x^4, x(1) = 1, x(2) = 2$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=0}^{\infty} [n(n-1)a_n x^n + n a_n x^n + a_n x^n] = x^6$$

$$[n(n-1) + n + 1]a_n x^n = x^6$$

$$(n^2 + 1)a_n x^n = x^6$$

$$37a_4 x^6 = x^6$$

$$a_4 = \frac{1}{37}$$

According to Wolfram the complete solution is as follows:

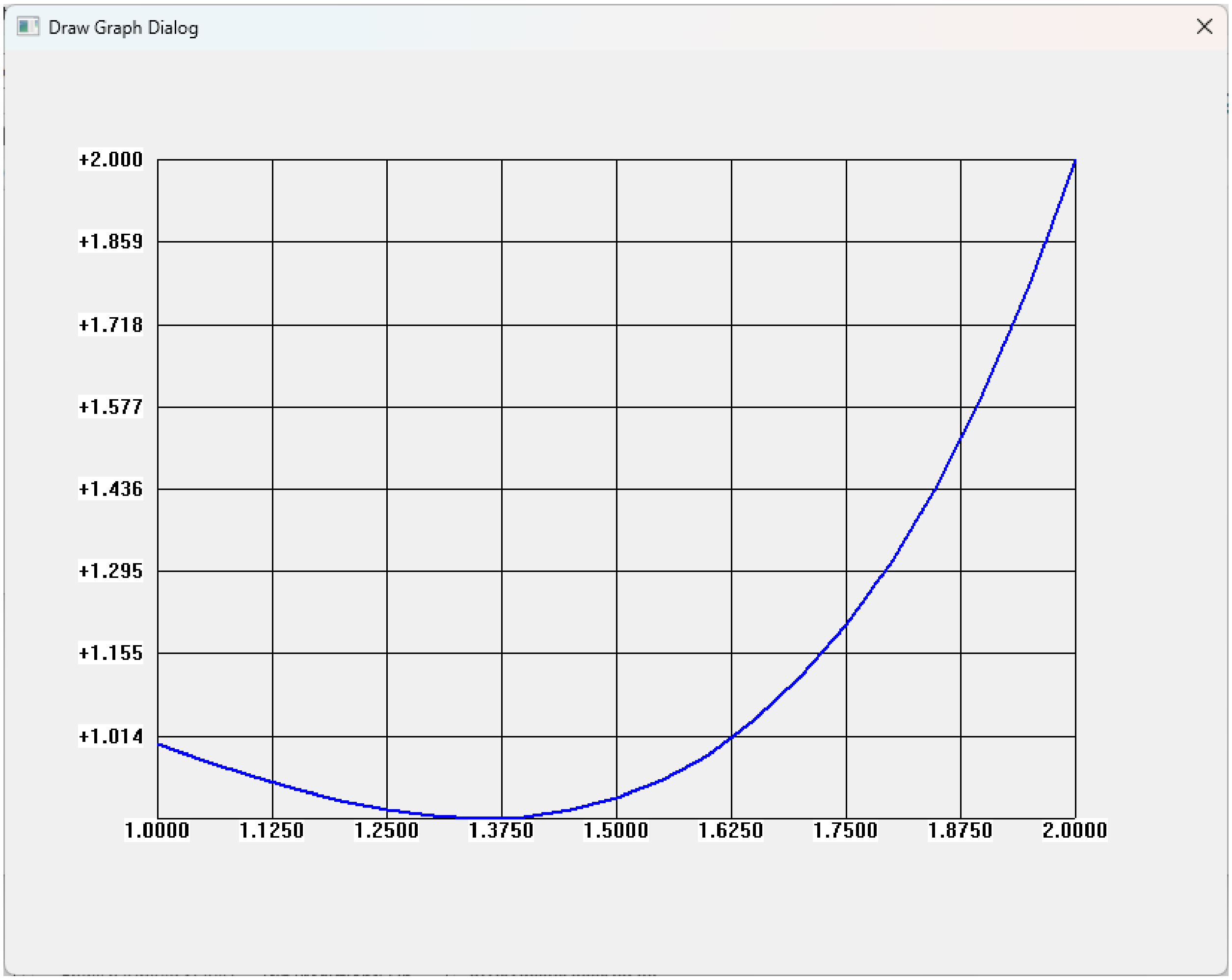
$$y(x) = c_2 \sin[\log(x)] + c_1 \cos[\log(x)] + \frac{x^6}{37}$$

$$y(1) = c_1 + \frac{1}{37} = 1 \therefore c_1 = 1 - \frac{1}{37}$$

$$y(2) = c_2 \sin[\log(2)] + c_1 \cos[\log(2)] + \frac{2^6}{37} = 2$$

$$c_2 \sin[\log(2)] = 2 - c_1 \cos[\log(2)] - \frac{64}{37}$$

n	x	y	exact	% error
0	1.00	1.0000000000	1.0000000000	0.0000000000
1	1.05	0.9717124556	0.9715354028	0.0182240136
2	1.10	0.9455565205	0.9452179058	0.0358239859
3	1.15	0.9222317673	0.9217476172	0.0525252434
4	1.20	0.9024727835	0.9018596073	0.0679902085
5	1.25	0.8870636752	0.8863384407	0.0818236627
6	1.30	0.8768512121	0.8760313408	0.0935892735
7	1.35	0.8727570100	0.8718603872	0.1028401879
8	1.40	0.8757890458	0.8748340405	0.1091641631
9	1.45	0.8870527230	0.8860582155	0.1122395227
10	1.50	0.9077616546	0.9067470683	0.1118929741
11	1.55	0.9392482860	0.9382336237	0.1081460234
12	1.60	0.9829744578	0.9819803393	0.1012360872
13	1.65	1.0405419796	1.0395896820	0.0916032176
14	1.70	1.1137032751	1.1128147731	0.0798427642
15	1.75	1.2043721424	1.2035701493	0.0666345121
16	1.80	1.3146346660	1.3139426754	0.0526651983
17	1.85	1.4467603086	1.4462026357	0.0385611849
18	1.90	1.6032132060	1.6028150284	0.0248423916
19	1.95	1.7866636822	1.7864510810	0.0119007597
20	2.00	2.0000000000	2.0000000000	0.0000000000


Draw Graph Dialog
+2.000
+1.859
+1.718
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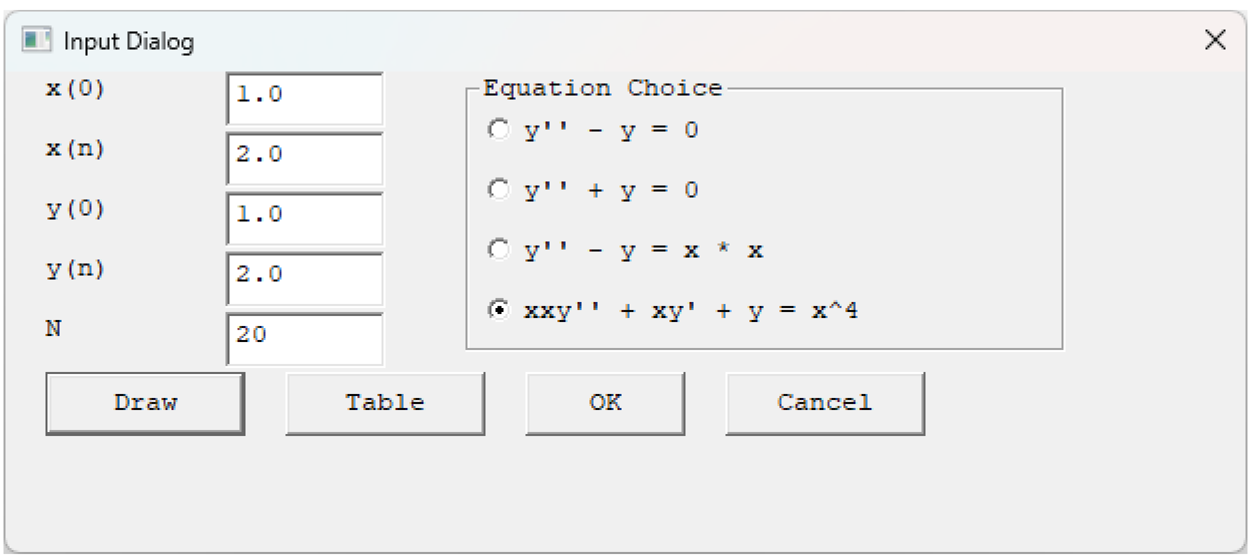
Input Dialog
x(0)
1.0
x(n)
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y(0)
1.0
y(n)
2.0
N
20
Equation Choice
y'' - y = 0
y'' + y = 0
y'' - y = x * x
xxy'' + xy' + y = x^4
Draw
Table
OK
Cancel