Blog Entry © Friday, August 1, 2025, by James Pate Williams, Jr. Numerically Solving a Two-Dimensional Elliptic Partial Differential Equation (PDE) Boundary Value Problem (BVP)

The PDE BVP is defined as follows:

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 2(x^2 + y^2), u(x,0) = 0, u(x,\pi) = \pi^2 x^2, \forall 0 \leq x \leq \pi, u(0,y) = 0, u(\pi,y) = \pi^2 y^2, \forall 0 \leq y \leq \pi$$

The analytical solution is:

$$u(x,y) = x^2 y^2$$

Checking the analytical solution:

$$\frac{\partial u(x,y)}{\partial x} = 2xy^2$$

$$\frac{\partial^2 u(x,y)}{\partial x^2} = 2y^2$$

$$\frac{\partial u(x,y)}{\partial y} = 2x^2 y$$

$$\frac{\partial^2 u(x,y)}{\partial x^2} = 2x^2$$

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 2y^2 + 2x^2 = 2(x^2 + y^2)$$

And the boundary values trivially satisfy the analytical solution.

We use the two functions found in ***A Numerical Library in C for Scientists and Engineers*** © 1995 by H. T. Lau, Ph.D. The C functions are from Chapter 5: Richardson and Elimination, pages 503 – 511. The testing main function is found in Worked Examples on pages 717 – 718. I changed all references to float, single precision, numbers to double, double precision, numbers. I also added rough runtime calculations.

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RICHARDSON and ELIMINATION deliver:

  K    DISCR[1]      DISCR[2]      RATECONV
 50  2.752502e-02   4.903205e-03   1.928270e-01

u[0][0] =  0.0000000000
u[1][1] =  0.0042015767
u[2][2] =  0.0974227224
u[3][3] =  0.5212655904
u[4][4] =  1.6776552085
u[5][5] =  4.1279711358
u[6][6] =  8.5922495937
u[7][7] = 15.9487091810
u[8][8] = 27.2337511694
u[9][9] = 43.6424333790

 dominant eigenvalue:  1.620289e-01

  7  1.521717e-07   4.527262e-08   1.692735e+00

Total number of iterations: 57
Rate of convergence with respect to
   the zeroth iterand of RICHARDSON:   3.770263e-01

time 1 = 0.001 seconds
time 2 = 0.000 seconds

D:\EllipticEq2d\x64\Release\EllipticEq2d.exe (process 9072)
exited with code 0 (0x0).
Press any key to close this window . . .
```