Vector Analysis by James Pate Williams, Jr. Exercises and Supplementary Problems from ***Introduction to Vector Analysis Fourth Edition*** © 1979 by Harry F. Davis and Arthur David Snider Selected Exercises from Chapter 1 Pages 48 and 50

1. Simplify

$$\left(\vec{A}\times\vec{B}\right)\cdot\vec{C}=\varepsilon_{ijk}A_iB_jC_k=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}\begin{matrix}\hat{\imath} & \hat{\jmath}\\ A_x & A_y\\ B_x & B_y\end{matrix}\cdot\vec{C}$$
$$=\left[\left(A_yB_z-A_zB_y\right)\hat{\imath}+\left(A_zB_x-A_xB_z\right)\hat{\jmath}+\left(A_xB_y-A_yB_x\right)\hat{k}\right]\cdot\vec{C}$$
$$=\left(A_yB_z-A_zB_y\right)C_x+\left(A_zB_x-A_xB_z\right)C_y+\left(A_xB_y-A_yB_x\right)C_z$$

2. Simplify

$$\left(\vec{A}\times\vec{B}\right)\cdot\left(\vec{C}\times\vec{D}\right)=\varepsilon_{ijk}A_iB_j\hat{e}_k\varepsilon_{lmn}C_lD_m\hat{e}_n=\varepsilon_{ijk}A_iB_j\varepsilon_{lmk}C_lD_m$$
$$=\varepsilon_{ijk}\varepsilon_{lmk}A_iB_jC_lD_m=\left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)A_iB_jC_lD_m$$
$$=A_iB_jC_iD_j-A_iB_jC_jD_i=A_iB_j\left(C_iD_j-C_jD_i\right)$$

3. Simplify

$$\left(\vec{A}\times\vec{B}\right)\cdot\left(\vec{B}\times\vec{C}\right)\times\left(\vec{C}\times\vec{A}\right)$$

First, we need to compute the second and third factors:

$$\left(\vec{B}\times\vec{C}\right)\times\left(\vec{C}\times\vec{A}\right)=\varepsilon_{ijk}B_iC_j\hat{e}_k\varepsilon_{lmn}C_lA_m\hat{e}_n$$

$$\vec{B}\times\vec{C}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}\begin{matrix}\hat{\imath} & \hat{\jmath}\\ B_x & B_y\\ C_x & C_y\end{matrix}$$
$$=\left(B_yC_z-B_zC_y\right)\hat{\imath}+\left(B_zC_x-B_xC_z\right)\hat{\jmath}+\left(B_xC_y-B_yC_x\right)\hat{k}$$

$$\vec{C}\times\vec{A}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ C_x & C_y & C_z\\ A_x & A_y & A_z\end{vmatrix}\begin{matrix}\hat{\imath} & \hat{\jmath}\\ C_x & C_y\\ A_x & A_y\end{matrix}$$
$$=\left(C_yA_z-C_zA_y\right)\hat{\imath}+\left(C_zA_x-C_xA_z\right)\hat{\jmath}+\left(C_xA_y-C_yA_x\right)\hat{k}$$

$$\left(\vec{B}\times\vec{C}\right)\times\left(\vec{C}\times\vec{A}\right)$$
$$=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ B_yC_z-B_zC_y & B_zC_x-B_xC_z & B_xC_y-B_yC_x\\ C_yA_z-C_zA_y & C_zA_x-C_xA_z & C_xA_y-C_yA_x\end{vmatrix}\begin{matrix}\hat{\imath} & \hat{\jmath}\\ B_yC_z-B_zC_y & B_zC_x-B_xC_z\\ C_yA_z-C_zA_y & C_zA_x-C_xA_z\end{matrix}$$
$$=\left[\left(B_zC_x-B_xC_z\right)\left(C_xA_y-C_yA_x\right)\right]\hat{\imath}+\left[\left(B_xC_y-B_yC_x\right)\left(C_yA_z-C_zA_y\right)\right]\hat{\jmath}$$
$$+\left[\left(B_yC_z-B_zC_y\right)\left(C_zA_x-C_xA_z\right)\right]\hat{k}-\left[\left(B_xC_y-B_yC_x\right)\left(C_zA_x-C_xA_z\right)\right]\hat{\imath}$$
$$-\left[\left(B_yC_z-B_zC_y\right)\left(C_xA_y-C_yA_x\right)\right]\hat{\jmath}-\left[\left(B_zC_x-B_xC_z\right)\left(C_yA_z-C_zA_y\right)\right]\hat{k}$$

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}\begin{matrix}\hat{\imath} & \hat{\jmath}\\ A_x & A_y\\ B_x & B_y\end{matrix}$$
$$=\left(A_yB_z-A_zB_y\right)\hat{\imath}+\left(A_zB_x-A_xB_z\right)\hat{\jmath}+\left(A_xB_y-A_yB_x\right)\hat{k}$$

$$(\vec{A} \times \vec{B}) \cdot (\vec{B} \times \vec{C}) \times (\vec{C} \times \vec{A})$$
$$= (A_y B_z - A_z B_y)(B_z C_x - B_x C_z)(C_x A_y - C_y A_x)$$
$$- (A_y B_z - A_z B_y)(B_x C_y - B_y C_x)(C_z A_x - C_x A_z)$$
$$+ (A_z B_x - A_x B_z)(B_x C_y - B_y C_x)(C_y A_z - C_z A_y)$$
$$- (A_z B_x - A_x B_z)(B_y C_z - B_z C_y)(C_x A_y - C_y A_x)$$
$$+ (A_x B_y - A_y B_x)(B_y C_z - B_z C_y)(C_z A_x - C_x A_z)$$
$$- (A_x B_y - A_y B_x)(B_z C_x - B_x C_z)(C_y A_z - C_z A_y)$$

Like any good computer scientist and software engineer, I will write a Win32 C/C++ console application to verify my formulas.

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Exercise 1 (a)  30
Exercise 1 (b)  -13
Exercise 1 (c)  5
Exercise 1 (d)  1
Exercise 2      5
Exercise 3      0
Exercise 4      0.666667
Exercise 5      1
Exercise 10     0.648886        0.648886

D:\VectorAnalysis\x64\Debug\VectorAnalysis.exe (process 12500)
exited with code 0 (0x0).
Press any key to close this window . . .
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Section 1.14 page 50

1. Derive the identity:

$$(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = [\vec{A}, \vec{B}, \vec{D}]\vec{C} - [\vec{A}, \vec{B}, \vec{C}]\vec{D}$$

$$\vec{A} \times \vec{B} = \varepsilon_{ijk} A_i B_j \hat{e}_k$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \begin{matrix} \hat{\imath} & \hat{\jmath} \\ A_x & A_y \\ B_x & B_y \end{matrix}$$
$$= (A_y B_z - A_z B_y)\hat{\imath} + (A_z B_x - A_x B_z)\hat{\jmath} + (A_x B_y - A_y B_x)\hat{k}$$

$$\vec{C} \times \vec{D} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ C_x & C_y & C_z \\ D_x & D_y & D_z \end{vmatrix} \begin{matrix} \hat{\imath} & \hat{\jmath} \\ C_x & C_y \\ D_x & D_y \end{matrix}$$
$$= (C_y D_z - C_z D_y)\hat{\imath} + (C_z D_x - C_x D_z)\hat{\jmath} + (C_x D_y - C_y D_x)\hat{k}$$

$$(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_yB_z - A_zB_y & A_zB_x - A_xB_z & A_xB_y - A_yB_x \\ C_yD_z - C_zD_y & C_zD_x - C_xD_z & C_xD_y - C_yD_x \end{vmatrix} \begin{matrix} \hat{\imath} & \hat{\jmath} \\ A_yB_z - A_zB_y & A_zB_x - A_xB_z \\ C_yD_z - C_zD_y & C_zD_x - C_xD_z \end{matrix}$$

$$[\vec{A}, \vec{B}, \vec{D}] = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ D_x & D_y & D_z \end{vmatrix}$$
$$= A_xB_yD_z + A_yB_zD_x + A_zB_xD_y - A_yB_xD_z - A_xB_zD_y - A_zB_yD_x$$

See computer output.

2. Derive the identity:

$$(\vec{A} \times \vec{B}) \times (\vec{B} \times \vec{C}) \times (\vec{C} \times \vec{A}) = [\vec{A}, \vec{B}, \vec{C}]^2$$

$$[\vec{A}, \vec{B}, \vec{C}] = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$
$$= A_xB_yC_z + A_yB_zC_x + A_zB_xC_y - A_yB_xC_z - A_xB_zC_y - A_zB_yC_x$$

See computer output.

3. Derive the identity:

$$\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = \vec{0}$$

See computer output.

4. Equation (1.29)

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

Derivation of the Equation (1.29):

$$\vec{A} \times (\vec{B} \times \vec{C}) = \varepsilon_{ijk} A_i (\vec{B} \times \vec{C})_j \hat{e}_k$$
$$(\vec{B} \times \vec{C})_j = -\varepsilon_{mnj} B_m C_n$$
$$\varepsilon_{ijk}\varepsilon_{mnj} = -\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}$$
$$\vec{A} \times (\vec{B} \times \vec{C}) = -\delta_{im}\delta_{jn} A_i B_m C_n \hat{e}_j + \delta_{in}\delta_{jm} A_i B_m C_n \hat{e}_j = -A_i B_i C_j \hat{e}_j + A_i C_i B_j \hat{e}_j$$
$$= (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

5. Simplify the acceleration vector:

$$\vec{a} = \vec{\omega} \times (\vec{\omega} \times \vec{R})$$
$$\vec{a} = \vec{\omega} \times (\vec{\omega} \times \vec{R}) = (\vec{\omega} \cdot \vec{R})\vec{\omega} - (\vec{\omega} \cdot \vec{\omega})\vec{R}$$

7. Simplify:

$$|\vec{A} \times \vec{B}|^2 + (\vec{A} \cdot \vec{B})^2 - |\vec{A}|^2 |\vec{B}|^2$$
$$|\vec{A} \times \vec{B}|^2 + (\vec{A} \cdot \vec{B})^2 - |\vec{A}|^2 |\vec{B}|^2 = |\vec{A}|^2 |\vec{B}|^2 (\sin\theta)^2 + |\vec{A}|^2 |\vec{B}|^2 (\cos\theta)^2 - |\vec{A}|^2 |\vec{B}|^2 = 0$$