Blog Entry © Monday, May 11, 2026, by James Pate Williams, Jr., Laplace Equation in a Solid Cylinder

Reference: laplacecylinder.pdf

Boundary Conditions from the web PDF:

$$u(r,\theta,H) = f(r,\theta)$$

$$u(r,\theta,0) = g(r,\theta)$$

$$u(a,\theta,z) = h(\theta,z)$$

Solution from the web PDF:

$$u(r,\theta,z) = u_1(r,\theta,z) + u_2(r,\theta,z) + u_3(r,\theta,z)$$

$$u_1(r,\theta,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n\left(\frac{z_{nm}r}{a}\right)\sinh\left(\frac{z_{nm}z}{a}\right)[a_{nm}\cos(n\theta) + b_{nm}\sin(n\theta)]$$

$$u_2(r,\theta,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n\left(\frac{z_{nm}r}{a}\right)\sinh\left(\frac{z_{nm}[H-z]}{a}\right)[c_{nm}\cos(n\theta) + d_{nm}\sin(n\theta)]$$

$$u_3(r,\theta,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} I_n\left(\frac{m\pi r}{H}\right)\sin\left(\frac{m\pi z}{H}\right)[e_{nm}\cos(n\theta) + f_{nm}\sin(n\theta)]$$

$$a_{0m} = \frac{\int_{-\pi}^{\pi}\int_0^a rf(r,\theta)J_0\left(\frac{z_{0m}r}{a}\right)drd\theta}{\pi a^2[J_1(z_{0m})]^2\sinh\left(\frac{z_{0m}H}{a}\right)} for\ n = 0, \forall m \geq 1$$

$$a_{nm} = \frac{2\int_{-\pi}^{\pi}\int_0^a rf(r,\theta)J_n\left(\frac{z_{nm}r}{a}\right)\cos(n\theta)\,drd\theta}{\pi a^2[J_{n+1}(z_{nm})]^2\sinh\left(\frac{z_{nm}H}{a}\right)} \forall n \geq 0, m \geq 1$$

$$b_{nm} = \frac{2\int_{-\pi}^{\pi}\int_0^a rf(r,\theta)J_n\left(\frac{z_{nm}r}{a}\right)\sin(n\theta)\,drd\theta}{\pi a^2[J_{n+1}(z_{nm})]^2\sinh\left(\frac{z_{nm}H}{a}\right)} \forall n \geq 0, m \geq 1$$

$$c_{0m} = \frac{\int_{-\pi}^{\pi}\int_0^a rg(r,\theta)J_0\left(\frac{z_{0m}r}{a}\right)drd\theta}{\pi a^2[J_1(z_{0m})]^2\sinh\left(\frac{z_{0m}H}{a}\right)} for\ n = 0, \forall m \geq 1$$

$$c_{nm} = \frac{2\int_{-\pi}^{\pi}\int_0^a rg(r,\theta)J_n\left(\frac{z_{nm}r}{a}\right)\cos(n\theta)\,drd\theta}{\pi a^2[J_{n+1}(z_{nm})]^2\sinh\left(\frac{z_{nm}H}{a}\right)} \forall n \geq 1, m \geq 1$$

$$d_{nm} = \frac{2\int_{-\pi}^{\pi}\int_0^a rg(r,\theta)\,J_n\left(\frac{z_{nm}r}{a}\right)\sin(n\theta)\,drd\theta}{\pi a^2[J_{n+1}(z_{nm})]^2\sinh\left(\frac{z_{nm}H}{a}\right)}\forall n \geq 1, m \geq 1$$

$$e_{0m} = \frac{\int_{-\pi}^{\pi}\int_0^H h(\theta,z)\sin\left(\frac{m\pi z}{H}\right)dzd\theta}{\pi H I_n\left(\frac{m\pi a}{H}\right)} for\ n = 0, \forall m \geq 1$$

$$e_{nm} = \frac{2\int_{-\pi}^{\pi}\int_0^H h(\theta,z)\sin\left(\frac{m\pi z}{H}\right)\cos(n\theta)\,dzd\theta}{\pi H I_n\left(\frac{m\pi a}{H}\right)}\forall n \geq 1, m \geq 1$$

$$f_{nm} = \frac{2\int_{-\pi}^{\pi}\int_0^H h(\theta,z)\sin\left(\frac{m\pi z}{H}\right)\sin(n\theta)\,dzd\theta}{\pi H I_n\left(\frac{m\pi a}{H}\right)}\forall n \geq 1, m \geq 1$$

We will use Monte-Carlo integration for the double integrals.

Suppose the boundary conditions are:

$$f(r,\theta) = r\cos(\theta)$$

$$g(r,\theta) = r\sin(\theta)$$

$$h(\theta,z) = z\cos(\theta)$$

This is a “hard” boundary value problem. A much simpler boundary value problem is by my own computations:

$$f(r,\theta) = g(r,\theta) = h(\theta,z) = 1$$

$$a_{0m} = \frac{\int_{-\pi}^{\pi}\int_0^a rf(r,\theta)\,J_0\left(\frac{z_{0m}r}{a}\right)drd\theta}{\pi a^2[J_1(z_{0m})]^2\sinh\left(\frac{z_{0m}H}{a}\right)} = \frac{2\pi\int_0^a r\,J_0\left(\frac{z_{0m}r}{a}\right)dr}{\pi a^2[J_1(z_{0m})]^2\sinh\left(\frac{z_{0m}H}{a}\right)} for\ n = 0, \forall m \geq 1$$

$$a_{nm} = \frac{2\int_{-\pi}^{\pi}\int_0^a rf(r,\theta)\,J_n\left(\frac{z_{nm}r}{a}\right)\cos(n\theta)\,drd\theta}{\pi a^2[J_{n+1}(z_{nm})]^2\sinh\left(\frac{z_{nm}H}{a}\right)}\forall n \geq 1, m \geq 1$$

$$\int_{-\pi}^{\pi}\cos(n\theta)\,d\theta = \sin(n\pi) - \sin(-n\pi) = 0$$

$$a_{nm} = 0, for\ n = 0, \forall m \geq 1$$

$$b_{nm} = \frac{2\int_{-\pi}^{\pi}\int_0^a rf(r,\theta)\, J_n\left(\frac{z_{nm}r}{a}\right)\sin(n\theta)\, drd\theta}{\pi a^2[J_{n+1}(z_{nm})]^2\sinh\left(\frac{z_{nm}H}{a}\right)} = \frac{2\int_{-\pi}^{\pi}\int_0^a r\, J_n\left(\frac{z_{nm}r}{a}\right)\sin(n\theta)\, drd\theta}{\pi a^2[J_{n+1}(z_{nm})]^2\sinh\left(\frac{z_{nm}H}{a}\right)} \forall n \geq 0, m \geq 1$$

$$\int_{-\pi}^{\pi}\sin(n\theta)d\theta = -\cos(n\pi) + \cos(-n\pi) = -(-1)^n - (-1)^n = (-1)^{n+1} + (-1)^{n+1} = 2(-1)^{n+1}\forall n \geq 1$$

$$c_{0m} = \frac{\int_{-\pi}^{\pi}\int_0^a rg(r,\theta)\, J_0\left(\frac{z_{0m}r}{a}\right) drd\theta}{\pi a^2[J_1(z_{0m})]^2\sinh\left(\frac{z_{0m}H}{a}\right)} = \frac{2\pi\int_0^a r\, J_0\left(\frac{z_{0m}r}{a}\right) dr}{\pi a^2[J_1(z_{0m})]^2\sinh\left(\frac{z_{0m}H}{a}\right)} = a_{0m}, for\ n = 0, \forall m \geq 1$$

$$c_{nm} = 0\forall n = 1, m \geq 1$$

$$e_{0m} = \frac{\int_{-\pi}^{\pi}\int_0^H h(\theta,z)\sin\left(\frac{m\pi z}{H}\right) dzd\theta}{\pi H I_n\left(\frac{m\pi a}{H}\right)} = \frac{2\pi\int_0^H \sin\left(\frac{m\pi z}{H}\right) dz}{\pi H I_n\left(\frac{m\pi a}{H}\right)} for\ n = 0, m \geq 1$$

$$\int_0^H \sin\left(\frac{m\pi z}{H}\right) dz = -\cos(m\pi) + \cos(0) = -(-1)^m + 1 = (-1)^{m+1} + 1$$

$$e_{nm} = 0\forall n \geq 1, m \geq 1$$

$$f_{nm} = \frac{2\int_{-\pi}^{\pi}\int_0^H \sin\left(\frac{m\pi z}{H}\right)\sin(n\theta)\, dzd\theta}{\pi H I_n\left(\frac{m\pi a}{H}\right)} = \frac{4[(-1)^{m+1}+1](-1)^n}{\pi H I_n\left(\frac{m\pi a}{H}\right)}\forall n \geq 1, m \geq 1$$

Testing Monte Carlo integration and Simpson's Rule by computing the volume of a sphere of unit radius:

$$dV = r^2\sin\vartheta drd\vartheta d\varphi$$

$$V = \int_0^1\int_0^\pi\int_0^{2\pi} r^2\sin\vartheta drd\vartheta d\varphi = \frac{1}{3}\cdot 2\cdot 2\pi = \frac{4\pi}{3}$$

$$V = \int_0^1\int_0^\pi\int_0^{2\pi} R(r)\,\mathrm{Theta}(\vartheta)\mathrm{Phi}(\varphi)\, drd\vartheta d\varphi$$

$$V = \int_0^1 r^2 dr \cdot \int_0^{\pi} \sin\vartheta d\vartheta \cdot \int_0^{2\pi} d\varphi$$

Test: Volume of a Unit Radius Sphere

Test: Volume of a Unit Radius Sphere

N	Monte Carlo	Simpson	Theory	% Error MC	% Error Simpso
1000	4.381867	4.188790	4.188790	4.609365	0.000000
2000	4.201789	4.188790	4.188790	0.310326	0.000000
3000	4.318732	4.188790	4.188790	3.102124	0.000000
4000	4.253616	4.188790	4.188790	1.547590	0.000000
5000	4.277932	4.188790	4.188790	2.128102	0.000000
6000	4.232255	4.188790	4.188790	1.037650	0.000000
7000	4.167355	4.188790	4.188790	0.511728	0.000000
8000	4.141694	4.188790	4.188790	1.124344	0.000000
9000	4.176295	4.188790	4.188790	0.298312	0.000000
10000	4.150947	4.188790	4.188790	0.903429	0.000000
11000	4.159377	4.188790	4.188790	0.702200	0.000000
12000	4.090295	4.188790	4.188790	2.351405	0.000000
13000	4.300369	4.188790	4.188790	2.663752	0.000000
14000	4.263418	4.188790	4.188790	1.781617	0.000000
15000	4.128071	4.188790	4.188790	1.449560	0.000000
16000	4.156034	4.188790	4.188790	0.781994	0.000000
17000	4.132496	4.188790	4.188790	1.343915	0.000000
18000	4.158316	4.188790	4.188790	0.727515	0.000000
19000	4.188970	4.188790	4.188790	0.004296	0.000000
20000	4.211845	4.188790	4.188790	0.550396	0.000000
21000	4.135459	4.188790	4.188790	1.273177	0.000000
22000	4.235605	4.188790	4.188790	1.117625	0.000000
23000	4.163814	4.188790	4.188790	0.596253	0.000000
24000	4.237008	4.188790	4.188790	1.151113	0.000000
25000	4.189090	4.188790	4.188790	0.007152	0.000000
26000	4.179301	4.188790	4.188790	0.226545	0.000000
27000	4.167549	4.188790	4.188790	0.507089	0.000000
28000	4.157917	4.188790	4.188790	0.737040	0.000000
29000	4.220656	4.188790	4.188790	0.760741	0.000000
30000	4.250849	4.188790	4.188790	1.481548	0.000000
31000	4.184940	4.188790	4.188790	0.091922	0.000000
32000	4.169418	4.188790	4.188790	0.462478	0.000000
33000	4.198402	4.188790	4.188790	0.229458	0.000000
34000	4.190288	4.188790	4.188790	0.035746	0.000000
35000	4.212102	4.188790	4.188790	0.556535	0.000000

Runtime in Seconds = 0.052000

Notice that the minimum percent error for the Monte Carlo Method occurs at N = 19,000 and is 0.004296%. We could improve the volume computation by using Gauss-Legendre quadrature. See accompanying Excel spreadsheet for more results.