Tensor Calculus Examples I © Monday, May, 4, 2026 by James Pate Williams, Jr.

Christoffel Symbols of the First Kind

Reference: ***Tensor Calculus*** © 1949 by J. L. Synge and A. Schild Equation 2.421 and Equation 2.422

$$[mn,r]=\frac{1}{2}\left(\frac{\partial a_{rm}}{\partial x^n}+\frac{\partial a_{rn}}{\partial x^m}-\frac{\partial a_{mn}}{\partial x^r}\right)$$

Christoffel Symbols of the Second Kind

$$\left\{ {r \atop mn} \right\} = a^{rs}[mn,s]$$

Find the Christoffel Symbols in Confocal Parabolic Coordinates

Parabolic coordinates - Wikipedia

$$x=\sigma\tau\cos\varphi$$

$$y=\sigma\tau\sin\varphi$$

$$z=\frac{1}{2}(\tau^2-\sigma^2)$$

$$ds^2=dx^2+dy^2+\ dz^2$$

$$dx=\tau\cos\varphi\,d\sigma+\sigma\cos\varphi d\tau-\sigma\tau\sin\varphi\,d\varphi$$

$$dy=\tau\sin\varphi\,d\sigma+\sigma\sin\varphi\,d\tau+\sigma\tau\cos\varphi\,d\varphi$$

$$dz=\tau d\tau-\sigma d\sigma$$

$$dx^2=\tau^2(\cos\varphi)^2d\sigma^2+\sigma^2(\cos\varphi)^2d\tau^2+\sigma^2\tau^2(\sin\varphi)^2d\varphi^2$$

$$dy^2=\tau^2(\sin\varphi)^2d\sigma^2+\sigma^2(\sin\varphi)^2d\tau^2+\sigma^2\tau^2(\cos\varphi)^2d\varphi^2$$

$$dz^2=\tau^2d\tau^2+\sigma^2d\sigma^2$$

$$\begin{aligned}ds^2&=\tau^2d\sigma^2+\sigma^2d\tau^2+\sigma^2\tau^2d\varphi^2+\tau^2d\tau^2+\sigma^2d\sigma^2\\&=(\sigma^2+\tau^2)d\sigma^2+(\sigma^2+\tau^2)d\tau^2+\sigma^2\tau^2d\varphi^2\end{aligned}$$

$$ds^2=a_{mn}x^mx^n,a_{mn}=0\forall m\neq n$$

$$a_{\sigma\sigma}=\sigma^2+\tau^2$$

$$a_{\tau\tau}=\sigma^2+\tau^2$$

$$a_{\varphi\varphi}=\sigma^2\tau^2$$

$$\frac{\partial a_{\sigma\sigma}}{\partial\sigma}=2\sigma$$

$$\frac{\partial a_{\sigma\sigma}}{\partial \tau} = 2\tau$$

$$\frac{\partial a_{\tau\tau}}{\partial \sigma} = 2\sigma$$

$$\frac{\partial a_{\tau\tau}}{\partial \tau} = 2\tau$$

$$\frac{\partial a_{\varphi\varphi}}{\partial \sigma} = 2\sigma\tau^2$$

$$\frac{\partial a_{\varphi\varphi}}{\partial \tau} = 2\sigma^2\tau$$

$$[\sigma\sigma,\sigma] = \frac{1}{2}\left(\frac{\partial a_{\sigma\sigma}}{\partial \sigma} + \frac{\partial a_{\sigma\sigma}}{\partial \sigma} - \frac{\partial a_{\sigma\sigma}}{\partial \sigma}\right) = -\sigma$$

$$[\sigma\sigma,\tau] = \frac{1}{2}\left(\frac{\partial a_{\tau\sigma}}{\partial \sigma} + \frac{\partial a_{\sigma\tau}}{\partial \sigma} - \frac{\partial a_{\sigma\sigma}}{\partial \tau}\right) = -\frac{1}{2}\frac{\partial a_{\sigma\sigma}}{\partial \tau} = -\tau$$

$$[\sigma\sigma,\varphi] = 0$$

$$[\tau\tau,\sigma] = \frac{1}{2}\frac{\partial a_{\tau\tau}}{\partial \sigma} = \sigma$$

$$[\tau\tau,\tau] = \frac{1}{2}\left(\frac{\partial a_{\tau\tau}}{\partial \tau} + \frac{\partial a_{\tau\tau}}{\partial \tau} - \frac{\partial a_{\tau\tau}}{\partial \tau}\right) = \tau$$

$$[\tau\tau,\varphi] = \frac{1}{2}\left(\frac{\partial a_{\varphi\tau}}{\partial \tau} + \frac{\partial a_{\tau\varphi}}{\partial \tau} - \frac{\partial a_{\sigma\sigma}}{\partial \varphi}\right) = [\sigma\sigma,\varphi] = 0$$

$$[\tau\tau,\sigma] = \frac{1}{2}\frac{\partial a_{\tau\tau}}{\partial \sigma} = \sigma$$

$$[\varphi\varphi,\sigma] = \frac{1}{2}\left(\frac{\partial a_{\sigma\varphi}}{\partial \varphi} + \frac{\partial a_{\sigma\varphi}}{\partial \varphi} - \frac{\partial a_{\varphi\varphi}}{\partial \sigma}\right) = \frac{1}{2}\frac{\partial a_{\varphi\varphi}}{\partial \sigma} = \sigma\tau^2$$

$$[\varphi\varphi,\tau] = \frac{1}{2}\left(\frac{\partial a_{\tau\varphi}}{\partial \varphi} + \frac{\partial a_{\tau\varphi}}{\partial \varphi} - \frac{\partial a_{\varphi\varphi}}{\partial \tau}\right) = \frac{1}{2}\frac{\partial a_{\varphi\varphi}}{\partial \tau} = \sigma^2\tau$$

$$[\varphi\varphi,\varphi] = 0$$

$$a^{rs} = \frac{1}{a_{rs}}, a_{mn} = 0 \forall m \neq n$$

$$\begin{Bmatrix} \sigma \\ \sigma\sigma \end{Bmatrix} = a^{\sigma\sigma}[\sigma\sigma,\sigma] = -\frac{\sigma}{\sigma^2+\tau^2}$$

$$\begin{Bmatrix} \tau \\ \sigma\sigma \end{Bmatrix} = a^{\tau\tau}[\sigma\sigma,\tau] = -\frac{\tau}{\sigma^2+\tau^2}$$

$$\begin{Bmatrix} \sigma \\ \tau\tau \end{Bmatrix} = a^{\sigma\sigma}[\tau\tau, \sigma] = \frac{\sigma}{\sigma^2 + \tau^2}$$

$$\begin{Bmatrix} \tau \\ \tau\tau \end{Bmatrix} = a^{\tau\tau}[\tau\tau, \tau] = \frac{\tau}{\sigma^2 + \tau^2}$$