Blog Entry © Sunday, May 17, 2026, by James Pate Williams, Jr. Derivation of the Time Independent Free Particle Schrödinger Equation in Confocal Parabolic Coordinates

References: [Parabolic coordinates - Wikipedia] [finite difference method for the Laplace operator in confocal parabolic coordinates]

[Parabolic coordinate PDE solutions]

$$x = \sigma\tau\cos\varphi$$

$$y = \sigma\tau\sin\varphi$$

$$z = \frac{1}{2}(\tau^2 - \sigma^2)$$

$$dx^2 = \tau^2(\cos\varphi)^2 d\sigma^2 + \sigma^2(\cos\varphi)^2 d\tau^2 + \sigma^2\tau^2(\sin\varphi)^2 d\varphi^2$$

$$dy^2 = \tau^2(\sin\varphi)^2 d\sigma^2 + \sigma^2(\sin\varphi)^2 d\tau^2 + \sigma^2\tau^2(\cos\varphi)^2 d\varphi^2$$

$$dz^2 = \sigma^2 d\sigma^2 + \tau^2 d\tau^2$$

$$\begin{aligned} ds^2 &= \tau^2 d\sigma^2 + \sigma^2 d\tau^2 + \sigma^2\tau^2 d\varphi^2 + \sigma^2 d\sigma^2 + \tau^2 d\tau^2 \\ &= (\sigma^2 + \tau^2) d\sigma^2 + (\sigma^2 + \tau^2) d\tau^2 + \sigma^2\tau^2 d\varphi^2 \end{aligned}$$

$$g_{\sigma\sigma} = g_{\tau\tau} = \sigma^2 + \tau^2$$

$$g_{\varphi\varphi} = \sigma^2\tau^2$$

$$h_\sigma = h_\tau = \sqrt{\sigma^2 + \tau^2}$$

$$h_\varphi = \sigma\tau$$

$$\begin{aligned} \nabla^2 &= \frac{1}{h_\sigma h_\tau h_\varphi}\left[\frac{\partial}{\partial\sigma}\left(\frac{h_\tau h_\varphi}{h_\sigma}\frac{\partial}{\partial\sigma}\right) + \frac{\partial}{\partial\tau}\left(\frac{h_\sigma h_\varphi}{h_\tau}\frac{\partial}{\partial\tau}\right) + \frac{\partial}{\partial\varphi}\left(\frac{h_\sigma h_\tau}{h_\varphi}\frac{\partial}{\partial\varphi}\right)\right] \\ &= \frac{1}{(\sigma^2+\tau^2)\sigma\tau}\left[\frac{\partial}{\partial\sigma}\left(\sigma\tau\frac{\partial}{\partial\sigma}\right) + \frac{\partial}{\partial\tau}\left(\sigma\tau\frac{\partial}{\partial\tau}\right) + \frac{\partial}{\partial\varphi}\left(\frac{\sigma^2+\tau^2}{\sigma\tau}\frac{\partial}{\partial\varphi}\right)\right] \\ &= \frac{1}{(\sigma^2+\tau^2)\sigma}\frac{\partial}{\partial\sigma}\left(\sigma\frac{\partial}{\partial\sigma}\right) + \frac{1}{(\sigma^2+\tau^2)\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial}{\partial\tau}\right) + \frac{1}{\sigma^2\tau^2}\frac{\partial^2}{\partial\varphi^2} \end{aligned}$$

The free particle Schrödinger equation in confocal parabolic coordinates is as follows:

$$-\frac{h^2}{8\pi^2 m}\left\{\frac{1}{(\sigma^2+\tau^2)\sigma}\frac{\partial}{\partial\sigma}\left(\sigma\frac{\partial}{\partial\sigma}\right) + \frac{1}{(\sigma^2+\tau^2)\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial}{\partial\tau}\right) + \frac{1}{\sigma^2\tau^2}\frac{\partial^2}{\partial\varphi^2}\right\}\psi(\sigma,\tau,\varphi) = 0$$

We now use separation of variables:

$$\psi(\sigma,\tau,\varphi) = \Sigma(\sigma,\tau)\Phi(\varphi)$$

$$-\frac{h^2}{8\pi^2 m}\left\{\frac{1}{(\sigma^2+\tau^2)\sigma}\frac{\partial}{\partial\sigma}\left(\sigma\frac{\partial}{\partial\sigma}\right)+\frac{1}{(\sigma^2+\tau^2)\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial}{\partial\tau}\right)+\frac{1}{\sigma^2\tau^2}\frac{\partial^2}{\partial\varphi^2}\right\}\Sigma(\sigma,\tau)\Phi(\varphi)$$
$$=-\frac{h^2}{8\pi^2 m}\left\{\frac{1}{(\sigma^2+\tau^2)\sigma}\frac{\partial}{\partial\sigma}\left(\sigma\frac{\partial\Sigma(\sigma,\tau)}{\partial\sigma}\right)+\frac{1}{(\sigma^2+\tau^2)\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial\Sigma(\sigma,\tau)}{\partial\tau}\right)\right\}\Phi(\varphi)$$
$$-\frac{h^2}{8\pi^2 m}\frac{1}{\sigma^2\tau^2}\frac{\partial^2\Phi(\varphi)}{\partial\varphi^2}\Sigma(\sigma,\tau)$$

Divide by the separate wavefunctions:

$$-\frac{h^2}{8\pi^2\mu}\frac{\sigma^2\tau^2}{\Sigma(\sigma,\tau)}\left\{\frac{1}{(\sigma^2+\tau^2)\sigma}\frac{\partial}{\partial\sigma}\left(\sigma\frac{\partial\Sigma(\sigma,\tau)}{\partial\sigma}\right)+\frac{1}{(\sigma^2+\tau^2)\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial\Sigma(\sigma,\tau)}{\partial\tau}\right)\right\}$$
$$=\frac{h^2}{8\pi^2\mu}\frac{1}{\Phi(\varphi)}\frac{\partial^2\Phi(\varphi)}{\partial\varphi^2}=-\lambda^2$$

We solve the easy Phi equation:

$$\frac{d^2\Phi(\varphi)}{d\varphi^2}+\frac{8\pi^2\mu\lambda^2}{h^2}\Phi(\varphi)=0$$

Set the Phi multiplication constant is as illustrated:

$$\alpha^2=\frac{8\pi^2\mu\lambda^2}{h^2}$$

$$\Phi(\varphi)=A\sin(\alpha\varphi)+B\cos(\alpha\varphi)$$

$$\frac{d\Phi}{d\varphi}=\alpha A\cos(\alpha\varphi)-\alpha B\sin(\alpha\varphi)$$

$$\frac{d^2\Phi}{d\varphi^2}=-\alpha^2 A\sin(\alpha\varphi)-\alpha^2 B\cos(\alpha\varphi)$$

The coupled equation is:

$$-\frac{h^2\sigma^2\tau^2}{8\pi^2 m}\left\{\frac{1}{(\sigma^2+\tau^2)\sigma}\frac{\partial}{\partial\sigma}\left(\sigma\frac{\partial\Sigma(\sigma,\tau)}{\partial\sigma}\right)+\frac{1}{(\sigma^2+\tau^2)\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial\Sigma(\sigma,\tau)}{\partial\tau}\right)\right\}+\lambda^2\Sigma(\sigma,\tau)=0$$
$$-\frac{h^2}{8\pi^2 m}\left\{\frac{\sigma\tau^2}{(\sigma^2+\tau^2)}\left[\sigma\frac{\partial^2\Sigma(\sigma,\tau)}{\partial\sigma^2}+\frac{\partial\Sigma(\sigma,\tau)}{\partial\sigma}\right]+\frac{\sigma^2\tau}{(\sigma^2+\tau^2)}\left[\tau\frac{\partial^2\Sigma(\sigma,\tau)}{\partial\tau^2}+\frac{\partial\Sigma(\sigma,\tau)}{\partial\tau}\right]\right\}$$
$$+\lambda^2\Sigma(\sigma,\tau)=0$$

$$\frac{\sigma\tau^2}{(\sigma^2+\tau^2)}\left[\sigma\frac{\partial^2\Sigma(\sigma,\tau)}{\partial\sigma^2}+\frac{\partial\Sigma(\sigma,\tau)}{\partial\sigma}\right]+\frac{\sigma^2\tau}{(\sigma^2+\tau^2)}\left[\tau\frac{\partial^2\Sigma(\sigma,\tau)}{\partial\tau^2}+\frac{\partial\Sigma(\sigma,\tau)}{\partial\tau}\right]-\frac{8\pi^2\mu\lambda^2}{h^2}\Sigma(\sigma,\tau)=0$$

$$\left[\frac{\partial^2\Sigma(\sigma,\tau)}{\partial\sigma^2}+\frac{1}{\sigma}\frac{\partial\Sigma(\sigma,\tau)}{\partial\sigma}\right]+\left[\frac{\partial^2\Sigma(\sigma,\tau)}{\partial\tau^2}+\frac{1}{\tau}\frac{\partial\Sigma(\sigma,\tau)}{\partial\tau}\right]-\frac{8\pi^2\mu\lambda^2(\sigma^2+\tau^2)}{h^2\sigma^2\tau^2}\Sigma(\sigma,\tau)=0$$

$$\frac{\partial^2\Sigma(\sigma,\tau)}{\partial\sigma^2}+\frac{1}{\sigma}\frac{\partial\Sigma(\sigma,\tau)}{\partial\sigma}+\frac{\partial^2\Sigma(\sigma,\tau)}{\partial\tau^2}+\frac{1}{\tau}\frac{\partial\Sigma(\sigma,\tau)}{\partial\tau}-\frac{8\pi^2\mu\lambda^2(\sigma^2+\tau^2)}{h^2\sigma^2\tau^2}\Sigma(\sigma,\tau)=0$$

We now compute the corresponding finite differences:

$$\frac{\partial^2\Sigma}{\partial\sigma^2}=\frac{\Sigma_{i+1,j}-2\Sigma_{i,j}+\Sigma_{i-1,j}}{(\Delta\sigma)^2}$$

$$\frac{\partial\Sigma}{\partial\sigma}=\frac{\Sigma_{i+1,j}-\Sigma_{i-1,j}}{2\Delta\sigma}$$

$$\frac{\partial^2\Sigma}{\partial\tau^2}=\frac{\Sigma_{i+1,j}-2\Sigma_{i,j}+\Sigma_{i-1,j}}{(\Delta\tau)^2}$$

$$\frac{\partial\Sigma}{\partial\tau}=\frac{\Sigma_{i+1,j}-\Sigma_{i-1,j}}{2\Delta\tau}$$

$$\frac{\Sigma_{i+1,j}-2\Sigma_{i,j}+\Sigma_{i-1,j}}{(\Delta\sigma)^2}+\frac{1}{\sigma}\frac{\Sigma_{i+1,j}-\Sigma_{i-1,j}}{2\Delta\sigma}+\frac{\Sigma_{i+1,j}-2\Sigma_{i,j}+\Sigma_{i-1,j}}{(\Delta\tau)^2}+\frac{1}{\tau}\frac{\Sigma_{i+1,j}-\Sigma_{i-1,j}}{2\Delta\tau}-\frac{8\pi^2\mu\lambda^2(\sigma^2+\tau^2)}{h^2\sigma^2\tau^2}\Sigma_{ij}=0$$

The previous equations have singular coordinates at the origin. Now suppose asymptotically near the origin we use the following coordinate transformation:

$$\sigma=e^u, u=\log\sigma, \tau=e^v, v=\log\tau$$

Using the chain rule differentiation formula, we obtain:

$$\frac{\partial\Sigma}{\partial\sigma}=\frac{\partial\Sigma}{\partial u}\frac{\partial u}{\partial\sigma}=\frac{1}{\sigma}\frac{\partial\Sigma}{\partial u}=e^{-u}\frac{\partial\Sigma}{\partial u}$$

$$\frac{\partial^2\Sigma}{\partial\sigma^2}=\frac{\partial}{\partial\sigma}\left(\frac{1}{\sigma}\frac{\partial\Sigma}{\partial u}\right)=-\frac{1}{\sigma^2}\frac{\partial\Sigma}{\partial u}+\frac{1}{\sigma}\frac{\partial^2\Sigma}{\partial\sigma\partial u}=-e^{-2u}\frac{\partial\Sigma}{\partial u}+e^{-u}\frac{\partial^2\Sigma}{\partial u^2}$$

$$e^{-u}\frac{\partial^2\Sigma}{\partial u^2}-e^{-2u}\frac{\partial\Sigma}{\partial u}+e^{-v}\frac{\partial^2\Sigma}{\partial v^2}-e^{-2v}\frac{\partial\Sigma}{\partial v}-\frac{8\pi^2\mu\lambda^2(e^{2u}+e^{2v})}{h^2e^{2u}e^{2v}}\Sigma=0$$

We seem to have good behavior as the coordinates u and v jointly approach infinity.

$$\lim_{\sigma,\tau\to\infty}\frac{\sigma^2+\tau^2}{\sigma^2\tau^2}=\lim_{\sigma,\tau\to\infty}\frac{2\sigma+2\tau}{2\sigma\tau^2+2\sigma^2\tau}=\lim_{\sigma,\tau\to\infty}\frac{4}{2\tau^2+4\sigma\tau+2\sigma^2+4\sigma\tau}=0$$

Now assume the eigenvalue lambda is zero:

$$\frac{\partial^2\Sigma(\sigma,\tau)}{\partial\sigma^2}+\frac{1}{\sigma}\frac{\partial\Sigma(\sigma,\tau)}{\partial\sigma}+\frac{\partial^2\Sigma(\sigma,\tau)}{\partial\tau^2}+\frac{1}{\tau}\frac{\partial\Sigma(\sigma,\tau)}{\partial\tau}=0$$

$$\frac{\partial^2\Sigma(\sigma,\tau)}{\partial\sigma^2}+\frac{1}{\sigma}\frac{\partial\Sigma(\sigma,\tau)}{\partial\sigma}=-\frac{\partial^2\Sigma(\sigma,\tau)}{\partial\tau^2}-\frac{1}{\tau}\frac{\partial\Sigma(\sigma,\tau)}{\partial\tau}=-n^2$$

$$\Sigma(\sigma,\tau)=S(\sigma)T(\tau)$$

$$\frac{1}{S}\left(\frac{d^2S}{d\sigma^2}+\frac{1}{\sigma}\frac{dS}{d\sigma}\right)+n^2=0$$

$$\frac{1}{T}\left(\frac{d^2T}{d\tau^2}+\frac{1}{\tau}\frac{dT}{d\tau}\right)-n^2=0$$

$$\frac{d^2S}{d\sigma^2}+\frac{1}{\sigma}\frac{dS}{d\sigma}+n^2S=0$$

$$\frac{d^2T}{d\tau^2}+\frac{1}{\tau}\frac{dT}{d\tau}-n^2T=0$$

$$\sigma^2\frac{d^2S}{d\sigma^2}+\sigma\frac{dS}{d\sigma}+n^2\sigma^2S=0$$

$$\tau^2\frac{d^2T}{d\tau^2}+\tau\frac{dT}{d\tau}-n^2\sigma^2T=0$$

$$S(\sigma)=\sum_{k=0}^{\infty}a_k\sigma^k$$

$$T(\tau)=\sum_{k=0}^{\infty}b_k\tau^k$$

$$\sum_{k=0}^{\infty}[k(k-1)a_k\sigma^{k-2}+ka_k\sigma^{k-1}+n^2\sigma^{k+2}a_k]=0$$

$$2\cdot 1a_2+2a_1+n^2\sigma^2a_0=0$$

$$3\cdot 2a_3+3a_2+n^2\sigma^3a_1=0$$

$$4\cdot 3a_4+4a_3+n^2\sigma^4a_2=0$$

$$k\cdot(k-1)a_k+ka_{k-1}+n^2\sigma^ka_{k-2}=0$$

$$a_k=-\frac{1}{k\cdot(k-1)}(ka_{k-1}+n^2\sigma^ka_{k-2})$$

Suppose the eigenvalue n is zero:

$$a_k=-\frac{1}{k-1}a_{k-1}$$

Let

$$a_2 = -a_1 = -1$$

$$a_3 = \frac{1}{2}$$

$$a_4 = -\frac{1}{3 \cdot 2}$$

$$a_5 = \frac{1}{4 \cdot 3 \cdot 2}$$

$$a_k = \frac{(-1)^{k+1}}{(k-1)!} \forall n \geq 2, a_1 = 1$$

$$b_k = \frac{(-1)^{k+1}}{(k-1)!} \forall n \geq 2, b_1 = 1$$