Blog Entry © Friday June 19, 2026, by James Pate Williams, Jr. Solution of a Simple Two-Dimensional System of Nonlinear Equations

Reference: ***Elementary Numerical Analysis Third Edition*** © 1980 S. D. Conte and Carl de Boor See Example 5.4, Example 5.5 and Exercise 5.2-3 page 222.

We solved the systems of nonlinear equations using Damped Newton's Method. See pages 216 – 222 of the preceding reference. First is Example 5.4, the Jacobian matrix is:

$$f'(\vec{x}) = \begin{bmatrix} -6x_1 & 2x_2 & 0 \\ 2x_1 + x_3 & -6x_2 & x_1 + 2x_3 \\ 0 & 2x_2 + 1 & -6x_3 \end{bmatrix}$$

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Example 5.4
x[ 0] = +2.5000000000e-01     x[ 1] = +5.0000000000e-01     x[
2] = +7.5000000000e-01

Approximate Jacobian Matrix

-2.0275057808e+00   +1.1705802764e+00   +0.0000000000e+00
+1.4774697570e+00   -3.5117408292e+00   +1.9411871235e+00
+0.0000000000e+00   +2.1705802764e+00   -4.8098099801e+00

Analytic Jacobian Matrix

-2.0275027809e+00   +1.1705792763e+00   +0.0000000000e+00
+1.4774687570e+00   -3.5117378290e+00   +1.9411861235e+00
+0.0000000000e+00   +2.1705792763e+00   -4.8098069800e+00

Jacobian Inverse Matrix

-7.2922100543e-01   -4.0876993644e-01   -1.8447051130e-01
-3.2386310312e-01   -5.6094734929e-01   -2.5314543735e-01
-1.3070764883e-01   -2.2639228871e-01   -3.1007531413e-01

Identity Matrix

+1.0000000000e+00   +1.1102230246e-16   +1.1102230246e-16
-1.1102230246e-16   +1.0000000000e+00   +0.0000000000e+00
+0.0000000000e+00   +0.0000000000e+00   +1.0000000000e+00

x[ 0] = +3.3791713015e-01     f[ 0] = -1.1102230246e-16
x[ 1] = +5.8528963816e-01     f[ 1] = -2.2204460493e-16
x[ 2] = +8.0163449666e-01     f[ 2] = +4.4408920985e-16

Iterations = 12
Runtime in Seconds = 0.000000
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The second example is Example 5.5 on pages 219 and 220:

$$f_1(x) = x_1 + 3\log|x_1| + x_2^2$$

$$f_2(x) = 2x_1^2 - x_1x_2 - 5x_1 + 1$$

The Jacobian matrix is:

$$f'(\vec{x}) = \begin{bmatrix} 1 + \dfrac{3}{x_1} & -2x_2 \\ 4x_1 - x_2 - 5 & -x_1 \end{bmatrix}$$

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Example 5.5
x[ 0] = +2.0000000000e+00     x[ 1] = +2.0000000000e+00

Approximate Jacobian Matrix

+3.7940079842e+00   +3.6597020823e+00
+1.9646749561e+00   -1.4276836477e+00

Analytic Jacobian Matrix

+3.1842353704e+00   +3.0499296728e+00
+2.0188782500e+00   -1.3734783534e+00

Jacobian Inverse Matrix

+1.3042345789e-01   +2.8961677717e-01
+1.9170967039e-01   -3.0237024610e-01

Identity Matrix

+1.0000000000e+00   +1.1102230246e-16
-1.1102230246e-16   +1.0000000000e+00

x[ 0] = +1.3734783534e+00     f[ 0] = -2.0872192863e-14
x[ 1] = -1.5249648364e+00     f[ 1] = -8.8817841970e-16

Iterations = 15
Runtime in Seconds = 0.000000
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Finally, we solve Exercise 5.2-3 on page 222. We calculate the partial derivatives then compute the Jacobian matrix and the inverse Jacobian matrix.

$$f(x,y) = \cos\left[\frac{x^2 - \sqrt{\sin(xy) + 3}}{4 + (xy)^2}\right] + \sin(3xy - 1) = 0.934$$

$$g(x,y) = e^{\{\cos[(xy)^3-3]\}} + \tan\left[\frac{x}{y}(0.08+\cos(x))\right] = 1.79$$

$$f(x) = f\big(g(x)\big) = \frac{df}{dg}\frac{dg}{dx}$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{d}{dx}(fg^{-1}) = \frac{df}{dx}g^{-1} - f\frac{dg}{dx}g^{-2}$$

$$f_1(x,y) = \frac{\partial}{\partial x}\left[\frac{x^2-\sqrt{\sin(xy)+3}}{4+(xy)^2}\right] = \frac{\partial}{\partial x}\left\{\left[x^2-\sqrt{\sin(xy)+3}\right][4+(xy)^2]^{-1}\right\}$$
$$= \left[2x - \frac{y\cos(xy)}{2\sqrt{\sin(xy)+3}}\right][4+(xy)^2]^{-1}$$
$$- 2xy^2\left[x^2-\sqrt{\sin(xy)+3}\right][4+(xy)^2]^{-2}$$

$$f_2(x,y) = -f_1(x,y)\sin\left[\frac{x^2-\sqrt{\sin(xy)+3}}{4+(xy)^2}\right]$$

$$f_3(x,y) = \frac{\partial}{\partial x}\sin(3xy-1) = 3y\cos(3xy-1)$$

$$f_4(x,y) = \frac{\partial}{\partial y}\left[\frac{x^2-\sqrt{\sin(xy)+3}}{4+(xy)^2}\right] = \frac{\partial}{\partial y}\left\{\left[x^2-\sqrt{\sin(xy)+3}\right][4+(xy)^2]^{-1}\right\}$$
$$= \left[-\frac{x\cos(xy)}{2\sqrt{\sin(xy)+3}}\right][4+(xy)^2]^{-1}$$
$$- 2x^2y\left[x^2-\sqrt{\sin(xy)+3}\right][4+(xy)^2]^{-2}$$

$$f_5(x,y) = -f_4(x,y)\sin\left[\frac{x^2-\sqrt{\sin(xy)+3}}{4+(xy)^2}\right]$$

$$f_6(x,y) = \frac{\partial}{\partial y}\sin(3xy-1) = 3x\cos(3xy-1)$$

$$g_1(x,y) = \frac{\partial}{\partial x}e^{\{\cos[(xy)^3-3]\}} = -3x^2y^3\sin[(xy)^3-3]\,e^{\{\cos[(xy)^3-3]\}}$$

$$g_2(x,y) = \frac{\partial}{\partial y}e^{\{\cos[(xy)^3-3]\}} = -3x^3y^2\sin[(xy)^3-3]\,e^{\{\cos[(xy)^3-3]\}}$$

$$\frac{d}{dx}\tan(x) = \frac{d}{dx}\frac{\sin(x)}{\cos(x)} = \frac{d}{dx}\{\sin(x)\,[\cos(x)]^{-1}\} = 1 + [\tan(x)]^2$$

$$g_3(x,y) = \frac{\partial}{\partial x}\left\{\tan\left[\frac{x}{y}(0.08+\cos(x))\right]\right\}$$
$$= \left[\frac{1}{y}(0.08+\cos(x)) - \frac{x}{y}\sin(x)\right]\left\{1+\left[\tan\left[\frac{x}{y}(0.08+\cos(x))\right]\right]^2\right\}$$

$$g_4(x,y) = \frac{\partial}{\partial y}\left\{\tan\left[\frac{x}{y}(0.08+\cos(x))\right]\right\}$$
$$= \left[-\frac{x}{y^2}\{0.08+\cos(x)\}\right]\left\{1+\left[\tan\left[\frac{x}{y}(0.08+\cos(x))\right]\right]^2\right\}$$

$$f'(\vec{x}) = \begin{bmatrix} f_2(x,y)+f_3(x,y) & f_5(x,y)+f_6(x,y) \\ g_1(x,y)+g_3(x,y) & g_2(x,y)+g_4(x,y) \end{bmatrix}$$

$$det = [f_2(x,y)+f_3(x,y)][g_2(x,y)+g_4(x,y)] - [f_5(x,y)+f_6(x,y)][g_1(x,y)+g_3(x,y)]$$

$$[f'(\vec{x})]^{-1} = \frac{1}{det}\begin{bmatrix} g_2(x,y)+g_4(x,y) & -f_5(x,y)-f_6(x,y) \\ -g_1(x,y)-g_3(x,y) & f_2(x,y)+f_3(x,y) \end{bmatrix}$$

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Exercise 5.2-3 Approximation
x[ 0] = +6.2562944426e-06     x[ 1] = +2.8179265725e-03

Approximate Jacobian Matrix

+1.7882463812e+00   +1.7717510277e+00
+3.1165543335e+00   -5.0929371902e+00

Analytic Jacobian Matrix

+1.7882463282e+00   +1.7717510113e+00
+3.1165540517e+00   -5.0929585774e+00

Jacobian Inverse Matrix

+3.4813540500e-01   +1.2111071442e-01
+2.1303677320e-01   -1.2223827919e-01

Identity Matrix

+1.0000000000e+00   +0.0000000000e+00
+0.0000000000e+00   +1.0000000000e+00

x[ 0] = +5.9093051110e-01     f[ 0] = -1.1102230246e-16
x[ 1] = +5.6276341441e-01     f[ 1] = -5.3290705182e-15

Iterations = 11
Runtime in Seconds = 0.079000
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Exercise 5.2-3 Analytic
x[ 0] = +5.0000000000e-01     x[ 1] = +5.0000000000e-01

Approximate Jacobian Matrix

+1.7882461212e+00   +1.7717511950e+00
+3.1165561596e+00   -5.0929421047e+00

Analytic Jacobian Matrix

+1.7882460681e+00   +1.7717511785e+00
+3.1165558777e+00   -5.0929634920e+00

Jacobian Inverse Matrix

+3.4813604526e-01   +1.2111032200e-01
+2.1303617035e-01   -1.2223785132e-01

Identity Matrix

+1.0000000000e+00   +0.0000000000e+00
+0.0000000000e+00   +1.0000000000e+00

x[ 0] = +5.9093051110e-01     f[ 0] = -5.5333709059e-08
x[ 1] = +5.6276341441e-01     f[ 1] = +6.2406620005e-07

Iterations = 4
Runtime in Seconds = 0.000000
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