References:
Elementary Numerical Analysis: An Algorithmic Approach © 1980 S. D. Conte and Carl de Boor
A Numerical Library in C for Scientists and Engineers © 1995 by H. T. Lau, PhD
Numerical Computation 2 Methods, Software, and Analysis © 1997 by Christoph W. Ueberhuber

Two Methods for Unconstrained Optimization

Our two techniques are:

1. Flemin
2. Praxis

We found functions implementing the two methods in the second reference previously mentioned. We optimize (minimize) the six functions enumerated below:

Ackley's function:

$$F_1 = -20e^{-0.2\sqrt{0.5(x^2+y^2)}} - e^{0.5\left[\cos(\pi^2 x)+\cos(\pi^2 y)\right]} + e + 20$$

$$\frac{\partial F_1}{\partial x} = -20\left[-0.2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot 2x\cdot\frac{1}{\sqrt{0.5(x^2+y^2)}}e^{-0.2\sqrt{0.5(x^2+y^2)}}\right] + \frac{1}{2}\pi^2\sin(\pi^2 x)e^{0.5\cos(\pi^2 x)}$$

$$\frac{\partial F_1}{\partial y} = -20\left[-0.2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot 2y\cdot\frac{1}{\sqrt{0.5(x^2+y^2)}}e^{-0.2\sqrt{0.5(x^2+y^2)}}\right] + \frac{1}{2}\pi^2\sin(\pi^2 y)e^{0.5\cos(\pi^2 y)}$$

Beale's function:

$$F_2(x,y) = (1.5 - x + xy)^2 + (2.25 - x + xy^2)^2 + (2.625 - x + xy^3)^2$$

$$\frac{\partial F_2}{\partial x} = 2(-1+y)(1.5 - x + xy) + (-1+y^2)(2.25 - x + xy^2) + (-1 + y^3)(2.625 - x + xy^3)$$

$$\frac{\partial F_2}{\partial y} = 2x(1.5 - x + xy) + 4xy(2.25 - x + xy^2) + 6xy^2(2.625 - x + xy^3)$$

Booth's function:

$$F_3(x,y) = (x + 2y - 7)^2 + (2x + y - 5)^2$$

$$\frac{\partial F_3}{\partial x} = 2(x + 2y - 7) + 4(2x + y - 5)$$

$$\frac{\partial F_3}{\partial y} = 4(x + 2y - 7) + 2(2x + y - 5)$$

Rosenbrock's function :

$$t = y - x^2$$

$$F_4(x, y) = 100t^2 + (1 - y)^2$$

$$\frac{\partial F_4}{\partial x} = 200 \frac{\partial t}{\partial x} t$$

$$\frac{\partial t}{\partial x} = -2x$$

$$\frac{\partial F_4}{\partial y} = 200 \frac{\partial t}{\partial y} t - 2(1 - y) = 200t - 2(1 - y)$$

$$\frac{\partial t}{\partial y} = 1$$

Hölder's table function:

$$F_5(x, y) = -\left|\sin(x)\cos(y) e^{\left|1 - \sqrt{x^2 + y^2}/\pi\right|}\right|$$

$$r = \frac{\sqrt{x^2 + y^2}}{\pi}$$

$$\frac{\partial r}{\partial x} = \frac{x}{\pi r}$$

$$\frac{\partial}{\partial x}|1 - r| = \frac{1 - r}{|1 - r|} \cdot \frac{\partial r}{\partial x} = \frac{1 - r}{|1 - r|} \cdot \frac{x}{\pi r}$$

$$\frac{\partial}{\partial x}\left|\sin(x)\cos(y) e^{\left|1 - \sqrt{x^2 + y^2}/\pi\right|}\right|$$

$$= -\frac{\sin(x)\cos(y) e^{\left|1 - \sqrt{x^2 + y^2}/\pi\right|}}{\left|\sin(x)\cos(y) e^{\left|1 - \sqrt{x^2 + y^2}/\pi\right|}\right|}$$

$$\cdot \left[\left|\cos(x)\cos(y) e^{\left|1 - \sqrt{x^2 + y^2}/\pi\right|} + \frac{1 - r}{|1 - r|} \cdot \frac{x}{\pi r} \cdot \sin(x)\cos(y) e^{\left|1 - \sqrt{x^2 + y^2}/\pi\right|}\right|\right]$$

$$\frac{\partial}{\partial y}\left|\sin(x)\cos(y)e^{\left|1-\sqrt{x^2+y^2}/\pi\right|}\right|$$
$$= -\frac{\sin(x)\cos(y)e^{\left|1-\sqrt{x^2+y^2}/\pi\right|}}{\left|\sin(x)\cos(y)e^{\left|1-\sqrt{x^2+y^2}/\pi\right|}\right|}$$
$$\cdot\left[-\left|\sin(x)\sin(y)\,e^{\left|1-\sqrt{x^2+y^2}/\pi\right|}+\frac{1-r}{|1-r|}\cdot\frac{y}{\pi r}\cdot\sin(x)\cos(y)e^{\left|1-\sqrt{x^2+y^2}/\pi\right|}\right|\right]$$

McCormick's function:

$$F_6(x,y) = \sin(x+y) + (x-y)^2 - 1.5x + 2.5y + 1$$
$$\frac{\partial F_6}{\partial x} = \cos(x+y) + 2(x-y) - 1.5$$
$$\frac{\partial F_6}{\partial y} = \cos(x+y) - 2(x-y) + 2.5$$

Next, we present the results discovered by our Win32 C console application.

```
== Method Menu ==
1 steepest descent
2 flemin
3 mininder
4 praxis
5 Exit
Method = 2
== Function Menu ==
1 Ackley
2 Beale
3 Booth
4 Rosenbrock
5 Holder Table
6 McCormick
Function = 2
test x        = +3.000000000000000e+00
test y        = +5.000000000000000e-01
test f(x, y)  = +0.000000000000000e+00
test dfdx     = -0.000000000000000e+00
test dfdy     = +0.000000000000000e+00
start x       = -1.764046143986328e+00
start y       = -4.365138706625569e+00
final x       = +2.999999999999969e+00
final y       = +4.999999999999916e-01
final f(x, y) = +1.669426890673966e-28
final dfdx    = -1.609823385706382e-15
final dfdy    = -3.363975764614293e-14
```

```
evaluations   = 4
runtime in s  =  0.004
== Method Menu ==
1 steepest descent
2 flemin
3 mininder
4 praxis
5 Exit
Method =

== Method Menu ==
1 steepest descent
2 flemin
3 mininder
4 praxis
5 Exit
Method = 2
== Function Menu ==
1 Ackley
2 Beale
3 Booth
4 Rosenbrock
5 Holder Table
6 McCormick
Function = 3
test x        = +1.000000000000000e+00
test y        = +3.000000000000000e+00
test f(x, y)  = +0.000000000000000e+00
test dfdx     = +0.000000000000000e+00
test dfdy     = +0.000000000000000e+00
start x       = -6.133915219580675e+00
start y       = +6.174810022278511e+00
final x       = +9.999999999999996e-01
final y       = +3.000000000000002e+00
final f(x, y) = +1.341063538875720e-29
final dfdx    = +1.065814103640150e-14
final dfdy    = +1.598721155460225e-14
evaluations   = 1
runtime in s  =  0.001
== Method Menu ==
1 steepest descent
2 flemin
3 mininder
4 praxis
5 Exit
Method =
```

```
== Method Menu ==
1 steepest descent
2 flemin
3 mininder
4 praxis
5 Exit
Method = 2
== Function Menu ==
1 Ackley
2 Beale
3 Booth
4 Rosenbrock
5 Holder Table
6 McCormick
Function = 4
test x        = +1.000000000000000e+00
test y        = +1.000000000000000e+00
test f(x, y)  = +0.000000000000000e+00
test dfdx     = -0.000000000000000e+00
test dfdy     = +0.000000000000000e+00
start x       = -2.994170964690085e+01
start y       = +7.919248023926511e+01
final x       = -1.000000000000005e+00
final y       = +1.000000000000012e+00
final f(x, y) = +6.476548031864507e-28
final dfdx    = +8.881784197001298e-13
final dfdy    = +4.689582056016661e-13
evaluations   = 2
runtime in s  =  0.003
== Method Menu ==
1 steepest descent
2 flemin
3 mininder
4 praxis
5 Exit
Method =

== Method Menu ==
1 steepest descent
2 flemin
3 mininder
4 praxis
5 Exit
Method = 4
== Function Menu ==
1 Ackley
2 Beale
```

```
3 Booth
4 Rosenbrock
5 Holder Table
6 McCormick
Function = 2
Normal Termination
test x        = +3.000000000000000e+00
test y        = +5.000000000000000e-01
test f(x, y)  = +0.000000000000000e+00
test dfdx     = -0.000000000000000e+00
test dfdy     = +0.000000000000000e+00
start x       = -3.174184392834254e+00
start y       = -3.006912442396313e+00
final x       = +3.000000000007882e+00
final y       = +4.999999999981969e-01
final f(x, y) = +3.355749324416854e-22
final dfdx    = +4.550068655195119e-11
final dfdy    = -1.733188592643812e-10
evaluations   = 116
runtime in s  =  0.005
== Method Menu ==
1 steepest descent
2 flemin
3 mininder
4 praxis
5 Exit
Method =

== Method Menu ==
1 steepest descent
2 flemin
3 mininder
4 praxis
5 Exit
Method = 4
== Function Menu ==
1 Ackley
2 Beale
3 Booth
4 Rosenbrock
5 Holder Table
6 McCormick
Function = 3
Normal Termination
test x        = +1.000000000000000e+00
test y        = +3.000000000000000e+00
test f(x, y)  = +0.000000000000000e+00
```

```
test dfdx     = +0.000000000000000e+00
test dfdy     = +0.000000000000000e+00
start x       = -9.974974822229683e+00
start y       = +1.271706289864802e+00
final x       = +1.000000000000000e+00
final y       = +3.000000000000000e+00
final f(x, y) = +0.000000000000000e+00
final dfdx    = +0.000000000000000e+00
final dfdy    = +0.000000000000000e+00
evaluations   = 119
runtime in s  =  0.001
== Method Menu ==
1 steepest descent
2 flemin
3 mininder
4 praxis
5 Exit
Method =

== Method Menu ==
1 steepest descent
2 flemin
3 mininder
4 praxis
5 Exit
Method = 4
== Function Menu ==
1 Ackley
2 Beale
3 Booth
4 Rosenbrock
5 Holder Table
6 McCormick
Function = 6
Normal Termination
test x        = -5.471900000000000e-01
test y        = -1.547190000000000e+00
test f(x, y)  = -1.913222954882274e+00
test dfdx     = +1.307911318515487e-05
test dfdy     = +1.307911318537691e-05
start x       = -1.493118076113163e+00
start y       = +9.450972014526808e-01
final x       = -5.471976961010339e-01
final y       = -1.547197696069247e+00
final f(x, y) = -1.913222954981000e+00
final dfdx    = -2.510178698234000e-07
final dfdy    = -2.508907224196832e-07
```

```
evaluations   = 122
runtime in s  =  0.001
== Method Menu ==
1 steepest descent
2 flemin
3 mininder
4 praxis
5 Exit
Method =
```

Back in 2015 I created a C# program that performs unconstrained optimization (minimization):

MainForm - Minimum (c) 2015 James Pate Williams, Jr.

PRAXIS Results

f#	Minimum	x	y	Evals
1	+8.813152E+000	-9.927672E-001	-3.970918E+000	91
2	+1.920702E-024	+3.000000E+000	+5.000000E-001	109
3	0.000000E+000	+1.000000E+000	+3.000000E+000	133
4	+2.147756E-023	+1.000000E+000	+1.000000E+000	551
5	-2.346576E+000	+4.971242E+000	+3.316439E+000	82
6	-1.913223E+000	-5.471976E-001	-1.547198E+000	115

Particle Swarm Optimization Results

f#	Minimum	x	y	Evals
1	+1.073662E-001	+1.243343E+000	+5.537199E-001	10000
2	+8.447032E-004	+1.119009E+000	+4.983479E-001	10000
3	+2.162476E-003	+2.486686E+000	+1.107440E+000	10000
4	+4.305409E-001	+2.486686E+001	+1.107440E+001	10000
5	-1.919670E+001	+2.486686E+000	+1.107440E+000	10000
6	-1.732814E+000	+9.946743E-001	+4.429759E-001	10000

Evolutionary Hill-Climber Results

f#	Minimum	x	y	Evals
1	+3.226415E-002	+7.846138E-003	-6.812959E-003	10000
2	+2.594447E-002	+3.521722E+000	+6.031769E-001	10000
3	+3.437165E-001	+7.450221E-001	+3.416910E+000	10000
4	+5.378876E-003	+9.273993E-001	+8.611088E-001	10000
5	-1.893379E+001	-8.021982E+000	-9.500724E+000	10000
6	-1.906871E+000	-4.780969E-001	-1.539332E+000	10000

FLEMIN Results

f#	Minimum	x	y	Evals
4	+4.513461E-018	+1.000000E+000	+1.000000E+000	109
6	-1.913223E+000	-5.471976E-001	-1.547198E+000	10