Blog Entry © Thursday, June 18, 2026, by James Pate Williams, Jr. Fourier Series Interpolation of Simple Polynomials

Reference: ***Fourier Series and Boundary Value Problems Third Edition*** © 1978 Ruel V. Churchill and James Ward Brown

Our Fourier Series is defined as follows:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)\,, x \in [-\pi\,, \pi)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx\,, n \geq 1, x \in [-\pi, \pi)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx, n \geq 1\,, x \in [-\pi, \pi)$$

Suppose our functions have the following form:

$$f(x) = \frac{x^e}{\pi^e}, e \in \{1,2,3,4\}$$

Now assume the exponent e is 3:

$$a_0 = \frac{1}{2\pi^4} \int_{-\pi}^{\pi} x^3 dx = \frac{1}{8\pi^4}(\pi^4 - \pi^4) = 0$$

$$a_n = \frac{1}{\pi^4} \int_{-\pi}^{\pi} x^3 \cos(nx)\, dx$$

$$\int \cos(nx)\, dx = \frac{1}{n} \sin(nx)$$

$$\frac{1}{n} \int \sin(nx)\, dx = -\frac{1}{n^2} \cos(nx)$$

$$-\frac{1}{n^2} \int \cos(nx)\, dx = -\frac{1}{n^3} \sin(nx)$$

$$-\frac{1}{n^3} \int \sin(nx)\, dx = \frac{1}{n^4} \cos(nx)$$

$$\frac{1}{n^4}\int \cos(nx)\,dx = \frac{1}{n^5}\sin(nx)$$

$$\int x^3\cos(nx)\,dx = -\frac{x^3}{n^2}\cos(nx) + \frac{3x^2}{n^3}\sin(nx) - \frac{6x}{n^4}\cos(nx) + \frac{6}{n^5}\sin(nx)$$

$$a_n = \frac{1}{\pi^4}\left[-\frac{\pi^3}{n^2}\cos(n\pi) - \frac{6\pi}{n^4}\cos(n\pi) + \frac{\pi^3}{n^2}\cos(n\pi) + \frac{6\pi}{n^4}\cos(n\pi)\right] = 0$$

$$b_n = \frac{1}{\pi^4}\int_{-\pi}^{\pi} x^3\sin(nx)\,dx$$

$$\int \sin(nx)\,dx = -\frac{1}{n}\cos(nx)$$

$$-\frac{1}{n}\int \cos(nx)\,dx = -\frac{1}{n^2}\sin(nx)$$

$$-\frac{1}{n^2}\int \sin(nx)\,dx = \frac{1}{n^3}\cos(nx)$$

$$\frac{1}{n^3}\int \cos(nx) = \frac{1}{n^4}\sin(nx)$$

$$\int x^3\sin(nx)\,dx = -\frac{x^3}{n}\cos(nx) + \frac{3x^2}{n^2}\sin(nx) + \frac{6x}{n^3}\cos(nx) - \frac{6}{n^4}\sin(nx)$$

$$b_n = \frac{1}{\pi^4}\left[-\frac{\pi^3}{n}\cos(n\pi) - \frac{\pi^3}{n}\cos(n\pi) + \frac{6\pi}{n^3}\cos(n\pi) + \frac{6\pi}{n^3}\cos(n\pi)\right]$$
$$= \frac{2(-1)^{n+1}}{n\pi} - \frac{12(-1)^{n+1}}{n^3\pi^3}$$

$$b_1 \sim +0.2496013591$$

$$b_2 \sim -0.2699325845$$

$$b_3 \sim +0.1978725755$$

$$b_4 \sim -0.1531077804$$

$$b_5 \sim +0.1242278072$$

$$b_6 \sim -0.1043115435$$

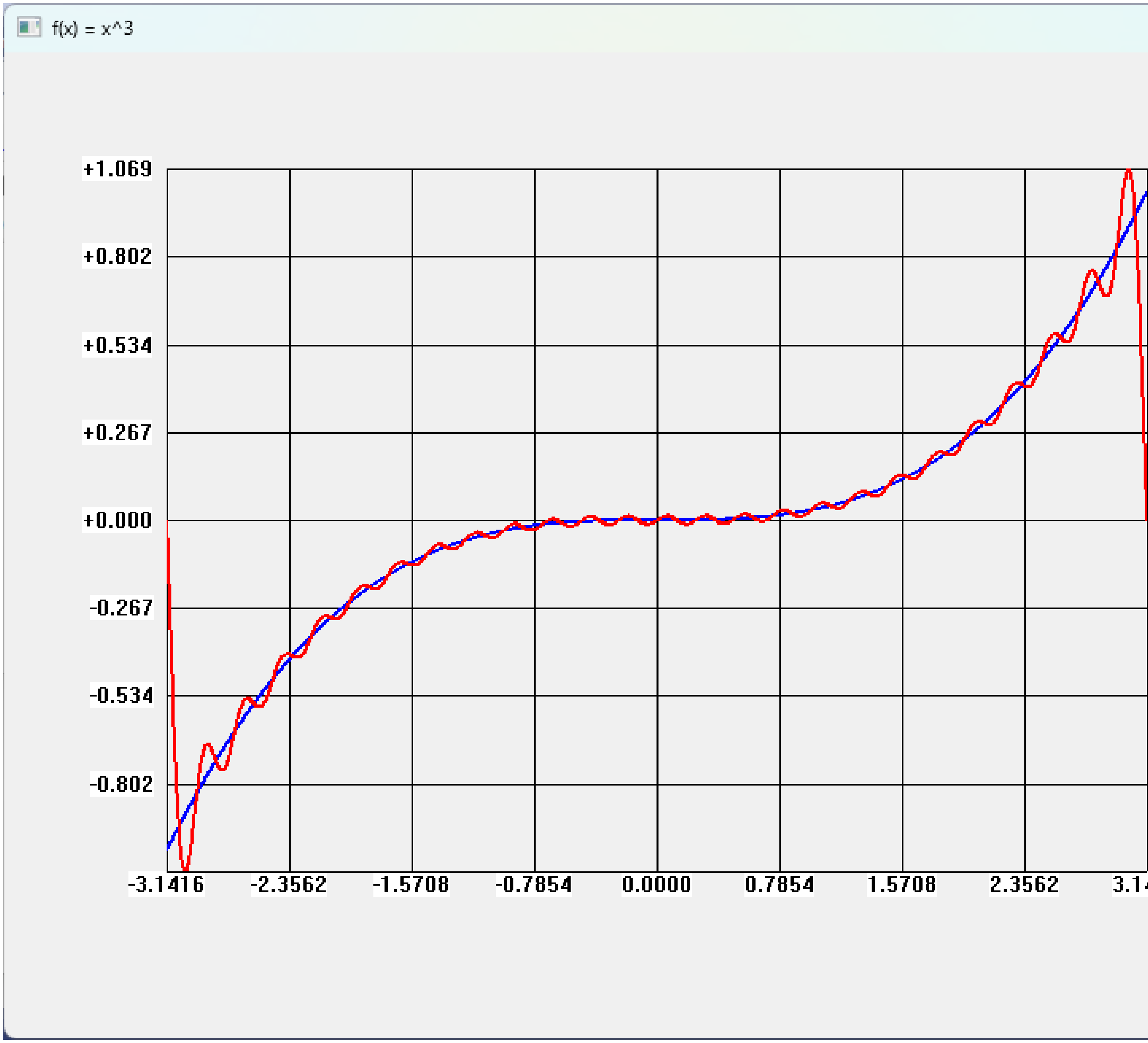
Using 1000 Simpson’s Rule steps and 15 Fourier terms we obtain the following table:

```
a[ 0] = +0.0000000000
a[ 1] = -0.0000000000    b[ 1] = +0.2496013592
```

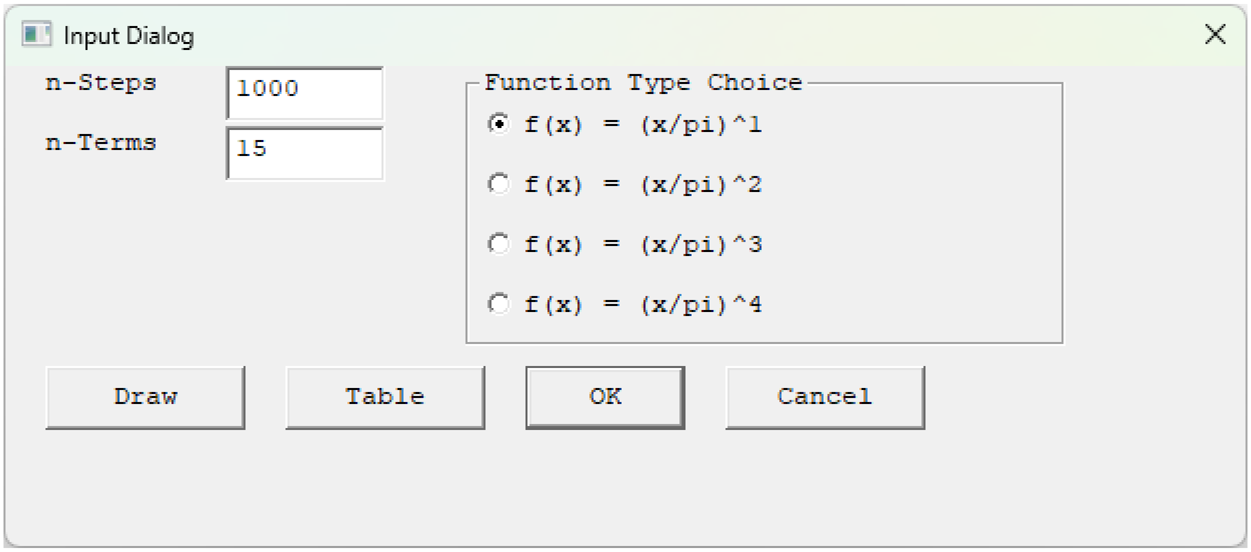
```
a[ 2] = +0.0000000000    b[ 2] = -0.2699325846
a[ 3] = -0.0000000000    b[ 3] = +0.1978725756
a[ 4] = +0.0000000000    b[ 4] = -0.1531077807
a[ 5] = -0.0000000000    b[ 5] = +0.1242278078
a[ 6] = +0.0000000000    b[ 6] = -0.1043115446
a[ 7] = -0.0000000000    b[ 7] = +0.0898173500
a[ 8] = +0.0000000000    b[ 8] = -0.0788215790
a[ 9] = -0.0000000000    b[ 9] = +0.0702046447
a[10] = +0.0000000000    b[10] = -0.0632749642
a[11] = -0.0000000000    b[11] = +0.0575837593
a[12] = +0.0000000000    b[12] = -0.0528276881
a[13] = -0.0000000000    b[13] = +0.0487946060
a[14] = +0.0000000000    b[14] = -0.0453318142
a[15] = -0.0000000000    b[15] = +0.0423266645
```

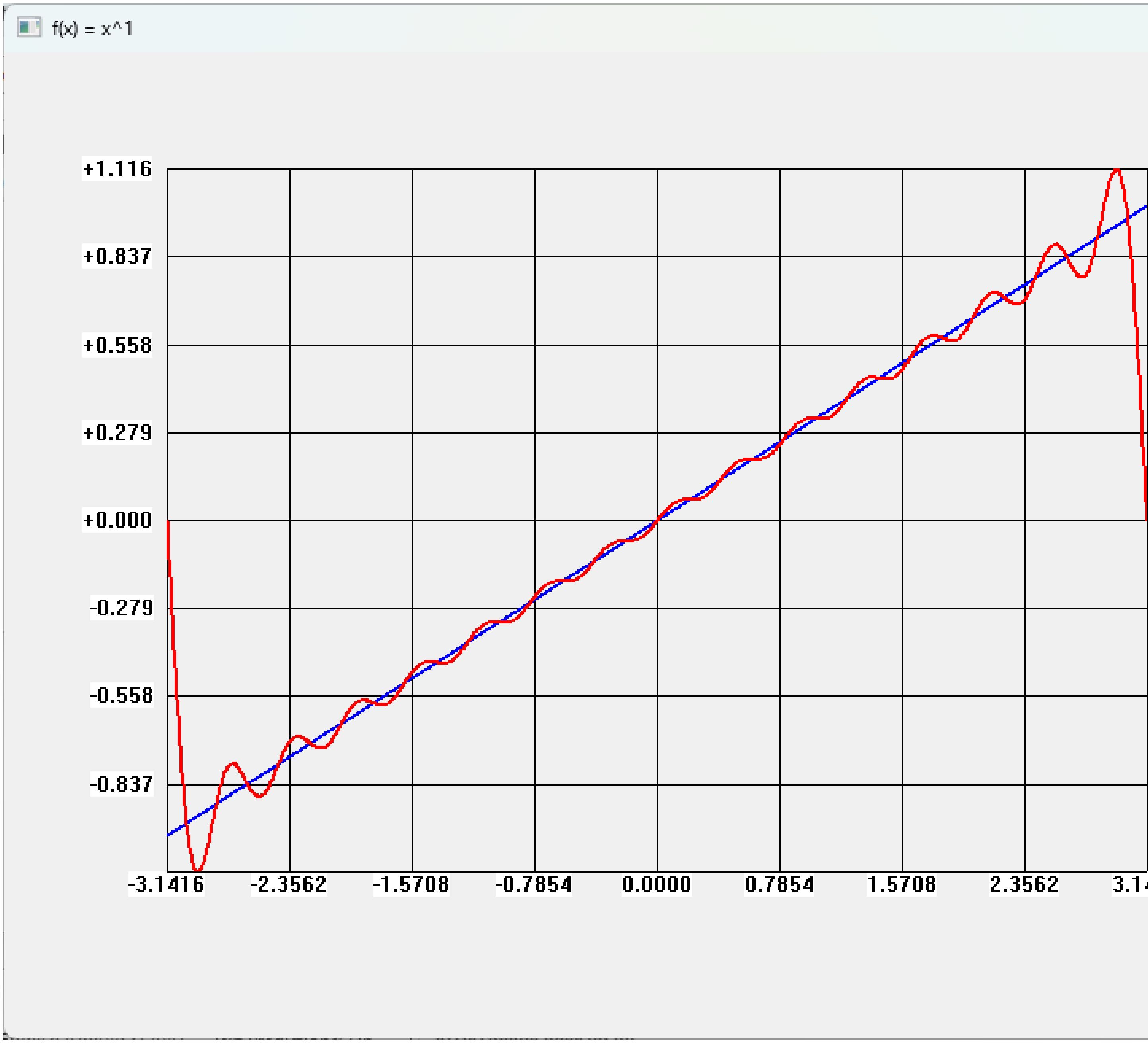
Utilizing 2000 Simpson's Rule steps and 25 Fourier terms yields the tabular results:

```
a[ 0] = +0.0000000000
a[ 1] = -0.0000000000    b[ 1] = +0.2496013592
a[ 2] = +0.0000000000    b[ 2] = -0.2699325845
a[ 3] = -0.0000000000    b[ 3] = +0.1978725755
a[ 4] = +0.0000000000    b[ 4] = -0.1531077804
a[ 5] = -0.0000000000    b[ 5] = +0.1242278072
a[ 6] = +0.0000000000    b[ 6] = -0.1043115436
a[ 7] = -0.0000000000    b[ 7] = +0.0898173483
a[ 8] = +0.0000000000    b[ 8] = -0.0788215764
a[ 9] = -0.0000000000    b[ 9] = +0.0702046411
a[10] = +0.0000000000    b[10] = -0.0632749592
a[11] = -0.0000000000    b[11] = +0.0575837526
a[12] = +0.0000000000    b[12] = -0.0528276793
a[13] = -0.0000000000    b[13] = +0.0487945948
a[14] = +0.0000000000    b[14] = -0.0453318001
a[15] = -0.0000000000    b[15] = +0.0423266472
a[16] = +0.0000000000    b[16] = -0.0396942503
a[17] = -0.0000000000    b[17] = +0.0373694492
a[18] = +0.0000000000    b[18] = -0.0353014059
a[19] = -0.0000000000    b[19] = +0.0334498813
a[20] = +0.0000000000    b[20] = -0.0317826141
a[21] = -0.0000000000    b[21] = +0.0302734403
a[22] = +0.0000000000    b[22] = -0.0289009195
a[23] = -0.0000000000    b[23] = +0.0276473159
a[24] = +0.0000000000    b[24] = -0.0264978325
a[25] = -0.0000000000    b[25] = +0.0254400271
```



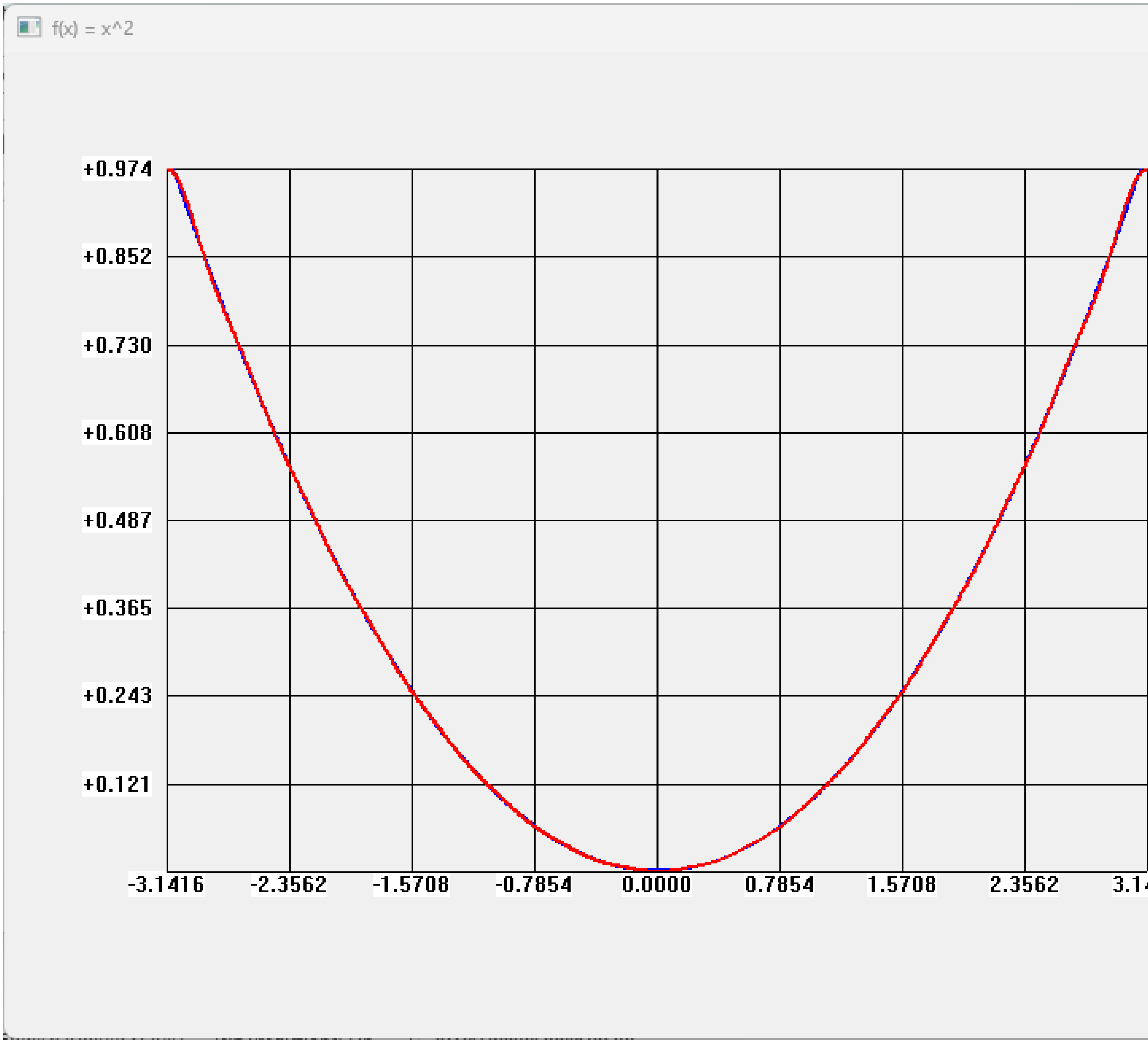
Next, we graph our trigonometric interpolations. Notice for odd exponents Gibb's Overshoot Phenomena are clearly illustrated:


Input Dialog
n-Steps
1000
n-Terms
15
Function Type Choice
f(x) = (x/pi)^1
f(x) = (x/pi)^2
f(x) = (x/pi)^3
f(x) = (x/pi)^4
Draw
Table
OK
Cancel



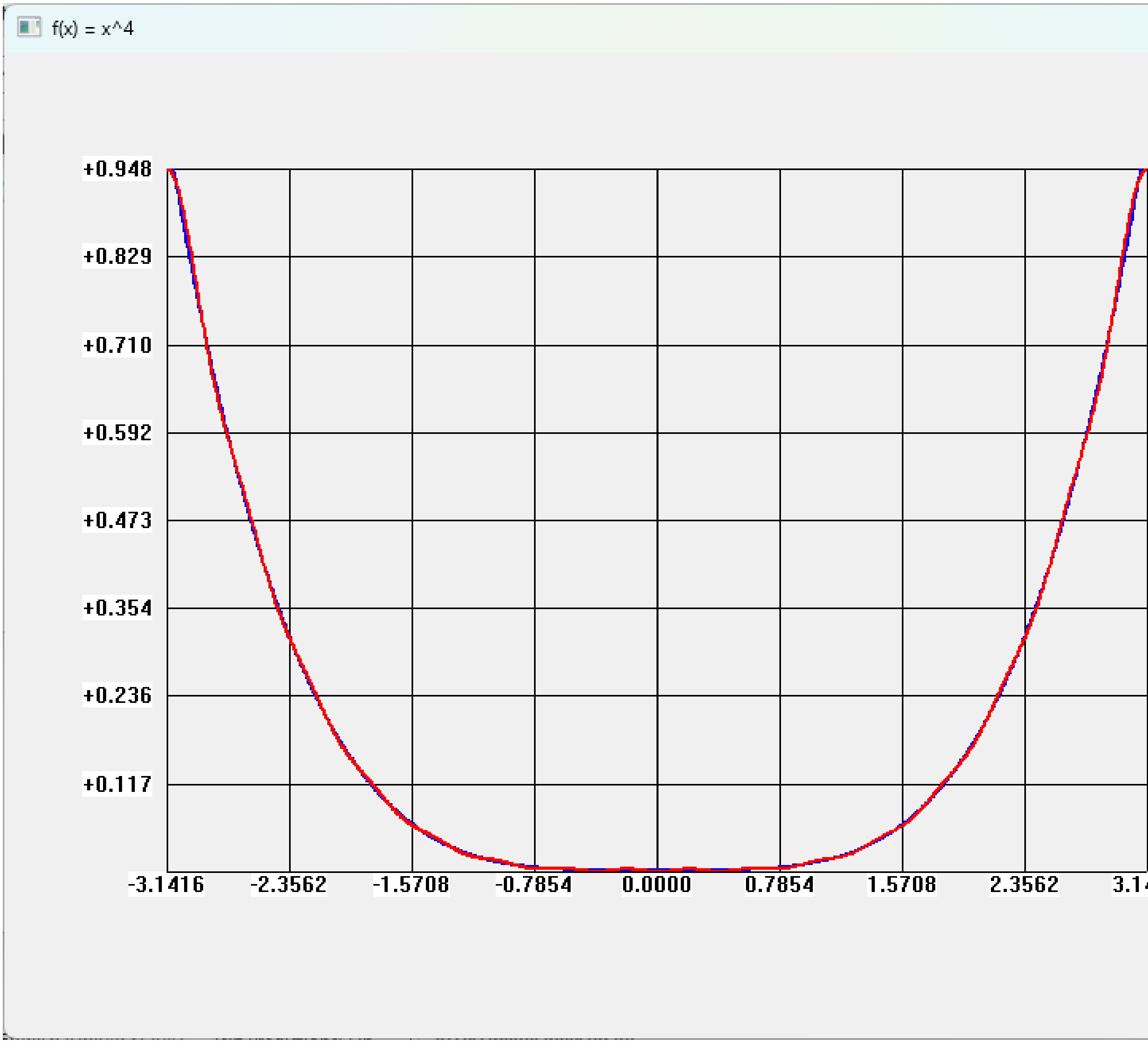

```
a[ 0] = +0.0000000000
a[ 1] = -0.0000000000    b[ 1] = +0.6366197724
a[ 2] = +0.0000000000    b[ 2] = -0.3183098862
a[ 3] = -0.0000000000    b[ 3] = +0.2122065909
a[ 4] = +0.0000000000    b[ 4] = -0.1591549434
a[ 5] = -0.0000000000    b[ 5] = +0.1273239552
a[ 6] = +0.0000000000    b[ 6] = -0.1061032966
a[ 7] = -0.0000000000    b[ 7] = +0.0909456837
a[ 8] = +0.0000000000    b[ 8] = -0.0795774744
a[ 9] = -0.0000000000    b[ 9] = +0.0707355343
```

```
a[10] = +0.0000000000    b[10] = -0.0636619828
a[11] = -0.0000000000    b[11] = +0.0578745321
a[12] = +0.0000000000    b[12] = -0.0530516572
a[13] = -0.0000000000    b[13] = +0.0489707638
a[14] = +0.0000000000    b[14] = -0.0454728560
a[15] = -0.0000000000    b[15] = +0.0424413368
```




```
a[ 0] = +0.3333333333
a[ 1] = -0.4052847346    b[ 1] = -0.0000000000
a[ 2] = +0.1013211836    b[ 2] = -0.0000000000
```

```
a[ 3] = -0.0450316371    b[ 3] = +0.0000000000
a[ 4] = +0.0253302957    b[ 4] = +0.0000000000
a[ 5] = -0.0162113891    b[ 5] = +0.0000000000
a[ 6] = +0.0112579089    b[ 6] = -0.0000000000
a[ 7] = -0.0082711165    b[ 7] = +0.0000000000
a[ 8] = +0.0063325733    b[ 8] = -0.0000000000
a[ 9] = -0.0050035144    b[ 9] = +0.0000000000
a[10] = +0.0040528463    b[10] = -0.0000000000
a[11] = -0.0033494593    b[11] = +0.0000000000
a[12] = +0.0028144758    b[12] = -0.0000000000
a[13] = -0.0023981327    b[13] = +0.0000000000
a[14] = +0.0020677772    b[14] = -0.0000000000
a[15] = -0.0018012631    b[15] = +0.0000000000
```




```
a[ 0] = +0.2000000000
a[ 1] = -0.3178023209    b[ 1] = +0.0000000000
a[ 2] = +0.1718444204    b[ 2] = -0.0000000000
a[ 3] = -0.0839797291    b[ 3] = +0.0000000000
a[ 4] = +0.0487357198    b[ 4] = -0.0000000000
a[ 5] = -0.0316343508    b[ 5] = +0.0000000000
a[ 6] = +0.0221355963    b[ 6] = -0.0000000000
a[ 7] = -0.0163369989    b[ 7] = +0.0000000000
a[ 8] = +0.0125448421    b[ 8] = -0.0000000000
a[ 9] = -0.0099319233    b[ 9] = +0.0000000000
```

```
a[10] = +0.0080564159    b[10] = -0.0000000000
a[11] = -0.0066652620    b[11] = +0.0000000000
a[12] = +0.0056051878    b[12] = -0.0000000000
a[13] = -0.0047790123    b[13] = +0.0000000000
a[14] = +0.0041227273    b[14] = -0.0000000000
a[15] = -0.0035927926    b[15] = +0.0000000000
```