Blog Entry © Saturday July 4, 2026, by James Pate Williams, Jr. Permutations Etc.

References: ***Quantum Mechanics Third Edition*** © 1968 by Leonard I. Schiff
Introduction to Vector Analysis Fourth Edition © 1979 by Harry F. Davis and Arthur David Snider
Molecular Symmetry and Spectroscopy © 1979 by Philip R. Bunker
Pate's Memory Which Is Sometimes Faulty

Permutations

There are n-factorial permutations of n numbers 1, 2, …, n.

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1	12
2	21

1	123
2	132
3	213
4	231
5	312
6	321

1	1234
2	1243
3	1324
4	1342
5	1423
6	1432
7	2134
8	2143
9	2341
10	2314
11	2134
12	2143
13	3124
14	3214
15	3241
16	3412
17	3421
18	3123
19	4123
20	4132
22	4213
22	4231
23	4312
24	4321
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Commutators

$$[A,B]=AB-BA$$

$$[A,BC]=ABC-BAC+BAC-BCA=[A,B]C+B[A,C]$$

$$p=-i\hbar\frac{\partial}{\partial x}$$

$$[x,p]f(x)=xpf(x)-pxf(x)=-i\hbar x\frac{\partial f}{\partial x}+i\hbar x\frac{\partial f}{\partial x}+i\hbar f(x)\therefore[x,p]=i\hbar$$

One-Dimensional Hamiltonian in Quantum Mechanics

$$H=T+V=\frac{1}{2m}p^2+V=\frac{i^2\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)$$

Laplacian Operator in General Three-Dimensional Orthogonal Coordinates

$$\nabla^2=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial u_3}\right)\right]$$

Next, we compute the Laplacian Operator in Spherical Coordinates:

$$x=r\sin(\vartheta)\cos(\varphi)$$

$$y=r\sin(\vartheta)\sin(\varphi)$$

$$z=r\cos(\vartheta)$$

Now calculate the line element:

$$ds^2=dx^2+dy^2+dz^2$$

$$dx^2=[\sin(\vartheta)]^2[\cos(\varphi)]^2dr^2+r^2[\cos(\vartheta)]^2[\cos(\varphi)]^2d\vartheta^2+r^2[\sin(\vartheta)]^2[\sin(\varphi)]^2d\varphi^2+\cdots$$

$$dy^2=[\sin(\vartheta)]^2[\sin(\varphi)]^2dr^2+r^2[\cos(\vartheta)]^2[\sin(\varphi)]^2d\vartheta^2+r^2[\sin(\vartheta)]^2[\cos(\varphi)]^2d\varphi^2+\cdots$$

$$dz^2=[\cos(\vartheta)]^2dr^2+r^2[\sin(\vartheta)]^2d\vartheta^2+\cdots$$

$$ds^2=dr^2+r^2d\vartheta^2+r^2[\sin(\vartheta)]^2d\varphi^2$$

The ellipsis stands for mixed differentials:

$$h_r=1,h_\vartheta=r,h_\varphi=r\sin(\vartheta)$$

$$\begin{aligned}\nabla^2&=\frac{1}{r^2\sin(\vartheta)}\left\{\frac{\partial}{\partial r}\left[r^2\sin(\vartheta)\frac{\partial}{\partial r}\right]+\frac{\partial}{\partial\vartheta}\left[\sin(\vartheta)\frac{\partial}{\partial\vartheta}\right]+\frac{1}{\sin(\vartheta)}\frac{\partial^2}{\partial\varphi^2}\right\}\\&=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\left[\sin(\vartheta)\frac{\partial}{\partial\vartheta}\right]+\frac{1}{r^2[\sin(\vartheta)]^2}\frac{\partial^2}{\partial\varphi^2}\end{aligned}$$

The Schrödinger equation for the hydrogen-like atom of Z = 1 atomic number is as follows:

$$-\frac{\hbar^2}{2\mu}\frac{1}{r^2\sin(\vartheta)}\left\{\frac{\partial}{\partial r}\left[r^2\sin(\vartheta)\frac{\partial\psi}{\partial r}\right]-\frac{\hbar^2}{2\mu}\frac{\partial}{\partial\vartheta}\left[\sin(\vartheta)\frac{\partial\psi}{\partial\vartheta}\right]-\frac{\hbar^2}{2\mu}\frac{1}{\sin(\vartheta)}\frac{\partial^2\psi}{\partial\varphi^2}\right\}-\frac{Ze^2\psi}{r}=E\psi$$

$$-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)-\frac{\hbar^2}{2\mu}\frac{1}{r^2\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\left[\sin(\vartheta)\frac{\partial\psi}{\partial\vartheta}\right]-\frac{\hbar^2}{2\mu}\frac{1}{r^2[\sin(\vartheta)]^2}\frac{\partial^2\psi}{\partial\varphi^2}-\frac{Ze^2\psi}{r}=E\psi$$

Where the reduced mass is in terms of the electron mass and the proton mass:

$$\frac{1}{\mu}=\frac{1}{m_e}+\frac{1}{m_p}$$

Suppose the following holds:

$$\psi(r,\vartheta,\varphi)=R(r)\Theta(\vartheta,\varphi)$$

The equation is partially separable:

$$-\frac{1}{r^2R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2\mu Ze^2}{\hbar^2 r}+\frac{2\mu E}{\hbar^2 r}=-\lambda^2$$

$$\frac{1}{r^2\Theta\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\left[\sin(\vartheta)\frac{\partial\Theta}{\partial\vartheta}\right]+\frac{1}{r^2\Theta[\sin(\vartheta)]^2}\frac{\partial^2\Theta}{\partial\varphi^2}=-\lambda^2$$

For the rest of the solution refer to Schiff or any other decent quantum mechanics or quantum chemistry textbook.