Blog Entry © Monday, August 3, 2026, by James Pate Williams, Jr., BA, BS, MSwE, PhD Free Particle Schrödinger Equation in Confocal Parabolic Coordinates

References: [Parabolic coordinates - Wikipedia] [Orthogonal coordinates - Wikipedia]

This is the corrected version of the work done while waiting for lunch to be served at Country's Barbecue on Sunday, August 2, 2026. I botched my standard paper airplane and accompanying handwritten text.

$$x = \sigma\tau\cos\varphi$$
$$y = \sigma\tau\sin\varphi$$

$$z = \frac{1}{2}(\tau^2 - \sigma^2)$$

$$dx = \tau\cos\varphi d\sigma + \sigma\cos\varphi\, d\tau - \sigma\tau\sin\varphi d\varphi$$

$$dy = \tau\sin\varphi d\sigma + \sigma\sin\varphi\, d\tau + \sigma\tau\cos\varphi d\varphi$$

$$dz = \tau d\tau - \sigma d\sigma$$

$$ds^2 = dx^2 + dy^2 + dz^2 = \tau^2 d\sigma^2 + \sigma^2 d\tau^2 + \sigma^2\tau^2 d\varphi^2 + \tau^2 d\tau^2 + \sigma^2 d\sigma^2$$
$$= (\sigma^2 + \tau^2)\, d\sigma^2 + (\sigma^2 + \tau^2)\, d\tau^2 + \sigma^2\tau^2 d\varphi^2$$

$$g_{\sigma\sigma} = \sigma^2 + \tau^2$$

$$g_{\tau\tau} = \sigma^2 + \tau^2$$

$$g_{\varphi\varphi} = \sigma^2\tau^2$$

$$h_\sigma = h_\tau = \sqrt{\sigma^2 + \tau^2}$$

$$h_\varphi = \sigma\tau$$

Suppose:

$$\nabla^2 = \frac{1}{h_\sigma h_\tau h_\varphi}\left[\frac{\partial}{\partial\sigma}\left(\frac{h_\tau h_\varphi}{h_\sigma}\frac{\partial}{\partial\sigma}\right) + \frac{\partial}{\partial\tau}\left(\frac{h_\sigma h_\varphi}{h_\tau}\frac{\partial}{\partial\tau}\right) + \frac{\partial}{\partial\varphi}\left(\frac{h_\sigma h_\tau}{h_\varphi}\frac{\partial}{\partial\varphi}\right)\right]$$
$$= \frac{1}{(\sigma^2 + \tau^2)\sigma\tau}\left[\frac{\partial}{\partial\sigma}\left(\sigma\tau\frac{\partial}{\partial\sigma}\right) + \frac{\partial}{\partial\tau}\left(\sigma\tau\frac{\partial}{\partial\tau}\right) + \frac{\partial}{\partial\varphi}\left(\frac{\sigma^2 + \tau^2}{\sigma\tau}\right)\frac{\partial}{\partial\varphi}\right]$$
$$= \frac{1}{(\sigma^2 + \tau^2)\sigma}\left(\sigma\frac{\partial^2}{\partial\sigma^2} + \frac{\partial}{\partial\sigma}\right) + \frac{1}{(\sigma^2 + \tau^2)\tau}\left(\tau\frac{\partial^2}{\partial\tau^2} + \frac{\partial}{\partial\tau}\right) + \frac{1}{\sigma^2\tau^2}\frac{\partial^2}{\partial\varphi^2}$$

The free particle non-relativistic Schrödinger equation is:

$$\nabla^2\psi = -\frac{2mE}{\hbar^2}\psi$$

Now vision the following partial variable separation:

$$\psi(\sigma,\tau,\varphi) = S(\sigma,\tau)\Phi(\varphi)$$

$$\frac{\sigma^2\tau^2}{(\sigma^2+\tau^2)\sigma S}\left(\sigma\frac{\partial^2 S}{\partial\sigma^2}+\frac{\partial S}{\partial\sigma}\right)+\frac{\sigma^2\tau^2}{(\sigma^2+\tau^2)\tau S}\left(\tau\frac{\partial^2 S}{\partial\tau^2}+\frac{\partial S}{\partial\tau}\right)+\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2}+\frac{2mE\sigma^2\tau^2}{\hbar^2}=0$$

$$\frac{\sigma^2\tau^2}{(\sigma^2+\tau^2)\sigma}\left(\sigma\frac{\partial^2 S}{\partial\sigma^2}+\frac{\partial S}{\partial\sigma}\right)+\frac{\sigma^2\tau^2}{(\sigma^2+\tau^2)\tau}\left(\tau\frac{\partial^2 S}{\partial\tau^2}+\frac{\partial S}{\partial\tau}\right)+\frac{2mE\sigma^2\tau^2}{\hbar^2}+\lambda^2 S=0$$

$$\frac{\partial^2\Phi}{\partial\varphi^2}+\lambda^2\Phi=0$$

The phi equation is very easy to solve:

$$\Phi(\varphi)=Ae^{i\lambda\varphi}\ or\ \Phi(\varphi)=B\cos(\lambda\varphi)+C\sin(\lambda\varphi)$$

We now verify the first form of the phi equation:

$$\Phi'(\varphi)=i\lambda Ae^{i\lambda\varphi}$$

$$\Phi''(\varphi)=i^2\lambda^2Ae^{i\lambda\varphi}=-\lambda^2Ae^{i\lambda\varphi}=-\lambda^2\Phi(\varphi)$$

The next step is to numerically solve the coupled two-dimensional equation using finite differences.