Blog Entry © Monday, September 7, 2026, by James Pate Williams, Jr. Solutions of Three Nonhomogeneous Second Order Linear Ordinary Differential Equation Boundary Value Problems and a Series Solution of the Bessel Function of the First Kind of Integer Eigenvalue (Order) of Zero

References: ***Boundary Value Problems Second Edition*** © 1979 by David L. Powers, ***Elementary Numerical Analysis an Algorithmic Approach Third Edition*** © 1980 by S. D. Conte and Carl de Boor

We use finite differences to solve the nonhomogeneous linear second order ordinary differential equation boundary value equation:

$$y''(x) + f(x)y'(x) + g(x)y(x) = q(x)$$

$$\frac{d^2y}{dx^2} = \frac{y(x+h) - 2y(x) + y(x-h)}{h^2}$$

We have a system of N – 1 by N – 1 linear system of equations:

$$\left(1 - \frac{h}{2}f_n\right)y_{n-1} + (-2 + h^2g_n)y_n + \left(1 + \frac{h}{2}f_n\right)y_{n+1} = h^2q_n \forall n \epsilon \{1, 2, \dots, N-1\}$$

$$x_n = x_0 + nh \forall n \epsilon \{1, 2, \dots, N-1\}$$

$$y_n = y(x_0 + nh) \forall n \epsilon \{0, 1, \dots, N\}$$

A Bessell equation and the boundary values are as follows:

$$x^2y''(x) + xy'(x) + x^2y(x) = 0, y(0) = 1, y(2) = \ 2.238907555e - 01$$

We expand the equation in terms of Infinite Power Series:

$$x^2y(x) = \sum_{n=0}^{\infty} a_n \mathrm{x}^{n+2}$$

$$a_0x^2 + a_1x^3 + a_2x^4 + a_3x^5 + a_4x^6 + a_5x^7 + a_6x^8 + a_7x^9 + a_8x^{10} + a_9x^{11} + a_{10}x^{12} + \cdots$$

$$xy'(x) = \sum_{n=0}^{\infty} na_n \mathrm{x}^n$$

$$a_1x^1 + 2a_2x^2 + 3a_3x^3 + 4a_4x^4 + 5a_5x^5 + 6a_6x^6 + 7a_7x^7 + 8a_8x^8 + 9a_9x^9 + 10a_{10}x^{10} + 11a_{11}x^{11} + 12a_{12}x^{12} + \cdots$$

$$x^2y''(x) = \sum_{n=0}^{\infty} n(n-1)a_nx^n$$

$$2a_2x^2 + 6a_3x^3 + 12a_4x^4 + 20a_5x^5 + 30a_6x^6 + 42a_7x^7 + 56a_8x^8 + 72a_9x^9 + 90a_{10}x^{10} + 110a_{11}x^{11} + 132a_{12}x^{12} + \cdots$$

$$a_0 + 2a_2 + 2a_2 = a_0 + 4a_2 = 0$$

$$a_1 = 0$$

$$a_2 = -\frac{1}{4}a_0$$

$$a_1 + 3a_3 + 6a_3 = a_1 + 9a_3 = 0$$

$$a_3 = 0$$

$$a_2 + 4a_4 + 12a_4 = a_2 + 16a_4 = 0$$

$$a_4 = -\frac{1}{16}a_2 = \frac{1}{64}a_0$$

$$a_3 + 5a_5 + 20a_5 = 0$$

$$a_5 = 0$$

$$a_4 + 6a_6 + 30a_6 = 0$$

$$a_6 = -\frac{1}{36}a_4 = -\frac{1}{36}\cdot\frac{1}{64}a_0$$

$$a_7 + 7a_9 + 42a_9 = 0$$

$$a_7 = 0$$

$$a_6 + 8a_8 + 56a_8 = 0$$

$$a_8 = -\frac{1}{64}a_6 = \frac{1}{64}\cdot\frac{1}{36}\cdot\frac{1}{64}a_0$$

$$a_{10} = -\frac{1}{100}\cdot a_8$$

$$a_{12} = -\frac{1}{144}\cdot a_{10}$$

Example 9.1

x	approximate	exact	percent error
0.05	0.0425650199	0.0425636361	0.0032510945
0.10	0.0852364523	0.0852337034	0.0032250823
0.15	0.1281209758	0.1281168994	0.0031817864
0.20	0.1713258018	0.1713204544	0.0031212927
0.25	0.2149589423	0.2149523998	0.0030437211
0.30	0.2591294802	0.2591218381	0.0029492237

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0.35 0.3039478417   0.3039392160   0.0028379848
0.40 0.3495260729   0.3495166002   0.0027102186
0.45 0.3959781192   0.3959679580   0.0025661688
0.50 0.4434201109   0.4434094420   0.0024061059
0.55 0.4919706528   0.4919596805   0.0022303262
0.60 0.5417511213   0.5417400745   0.0020391497
0.65 0.5928859677   0.5928751008   0.0018329179
0.70 0.6455030290   0.6454926237   0.0016119920
0.75 0.6997338478   0.6997242144   0.0013767504
0.80 0.7557140013   0.7557054800   0.0011275869
0.85 0.8135834397   0.8135764031   0.0008649083
0.90 0.8734868368   0.8734816908   0.0005891322
0.95 0.9355739510   0.9355711378   0.0003006851
Exercise 9.1-1
 x   approx
0.25 0.2943
0.50 0.5702
0.75 0.8104
Exercise 9.1-3
 x    approximate
0.10 0.8740659265
0.20 0.7426778048
0.30 0.6104985785
0.40 0.4821226931
0.50 0.3618976003
0.60 0.2537625354
0.70 0.1611133721
0.80 0.0867002548
0.90 0.0325621826
Exercise 9.1-4
 x    approximate
0.06 0.1432584454
0.12 0.2693660500
0.19 0.3801067083
0.25 0.4771109410
0.31 0.5618679443
0.38 0.6357367415
0.44 0.6999565010
0.50 0.7556560811
0.56 0.8038628574
0.62 0.8455108848
0.69 0.8814484424
0.75 0.9124450064
0.81 0.9391976915
0.88 0.9623372015
0.94 0.9824333236
Bessel Equation of Order Zero
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Approximate Finite Difference Solution

x	approximate
0.06	0.9992526934
0.12	0.9964013707
0.19	0.9915768227
0.25	0.9848107054
0.31	0.9761286880
0.38	0.9655588532
0.44	0.9531335645
0.50	0.9388900147
0.56	0.9228703752
0.62	0.9051217860
0.69	0.8856962701
0.75	0.8646506023
0.81	0.8420461483
0.88	0.8179486792
0.94	0.7924281689
1.00	0.7655585728
1.06	0.7374175941
1.12	0.7080864349
1.19	0.6776495369
1.25	0.6461943104
1.31	0.6138108546
1.38	0.5805916677
1.44	0.5466313515
1.50	0.5120263079
1.56	0.4768744308
1.62	0.4412747932
1.69	0.4053273316
1.75	0.3691325281
1.81	0.3327910906
1.88	0.2964036344
1.94	0.2600703636

Bessel Equation of Order Zero

Approximate Infinite Series Solution

x	approximate
0.06	0.9990236759
0.12	0.9960975630
0.19	0.9912302306
0.25	0.9844359293
0.31	0.9757345455
0.38	0.9651515361
0.44	0.9527178460
0.50	0.9384698072
0.56	0.9224490211
0.62	0.9047022230
0.69	0.8852811304

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0.75 0.8642422752
0.81 0.8416468202
0.88 0.8175603610
0.94 0.7920527132
1.00 0.7651976866
1.06 0.7370728462
1.12 0.7077592621
1.19 0.6773412472
1.25 0.6459060853
1.31 0.6135437507
1.38 0.5803466179
1.44 0.5464091654
1.50 0.5118276724
1.56 0.4766999109
1.62 0.4411248326
1.69 0.4052022535
1.75 0.3690325362
1.81 0.3327162709
1.88 0.2963539566
1.94 0.2600456836
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