

Laplace (Potential) Equation in Three Different Ecosystems by James Pate Williams, Jr., BA, BS, MSwE, PhD

Reference: **Boundary Value Problems Second Edition © 1979** by David L. Powers

First, we compute the exact and approximate potential in a rectangle using Fourier Series and finite elements. In this case the Laplace Equation is:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \forall x \in (0, a), \forall y \in (0, b)$$

$$u(x, 0) = f_1(x), x \in (0, a)$$

$$u(x, b) = f_2(x), x \in (0, a)$$

$$u(0, y) = 0, \forall y \in (0, b)$$

$$u(a, y) = 0, \forall y \in (0, b)$$

The finite elements required are

$$\frac{\partial^2 u}{\partial x^2} \sim \frac{u(x + \Delta x, y) - 2u(x, y) + u(x - \Delta x, y)}{(\Delta x)^2}$$

$$\frac{\partial^2 u}{\partial y^2} \sim \frac{u(x, y + \Delta y) - 2u(x, y) + u(x, y - \Delta y)}{(\Delta y)^2}$$

Suppose that delta x and delta y are equal then the approximate Laplace Equation is:

$$u(x, y) = 0.25[u(x + \Delta x, y) + u(x - \Delta x, y) + u(x, y + \Delta y) + u(x, y - \Delta y)]$$

Furthermore, suppose the boundary values are:

$$u(0, y) = 0, \forall y \in (0, 1)$$

$$u(1, y) = 0, \forall y \in (0, 1)$$

$$u(x, 0) = f(x), \forall x \in (0, 1)$$

$$u(x, 1) = f(x), \forall x \in (0, 1)$$

$$f(x) = \begin{cases} 2x, \forall x \in \left(0, \frac{1}{2}\right) \\ 2 - 2x, \forall x \in \left[\frac{1}{2}, 1\right) \end{cases}$$

The exact solution is found to be:

$$u(x, y) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n^2} \frac{\sinh\left[\left(\frac{n\pi}{a}\right)y\right] + \sinh\left[\left(\frac{n\pi}{a}\right)(b-y)\right]}{\sinh\left[\left(\frac{n\pi}{a}\right)b\right]} \sin\left(\frac{n\pi x}{a}\right)$$

Now suppose $a = b = 1$.

Potential2d

File Help

Nx

Ny

Nt

Ni

Potential Options

☒ Potential in a Rectangle

☐ Potential in a Slot

☐ Potential in a Disk

Compute

Clear

x	y	Exact	Approx	% Error
0.250000	0.250000	0.296484	0.273071	7.896825
0.250000	0.500000	0.227281	0.209259	7.929349
0.250000	0.750000	0.296484	0.291931	1.535654
0.500000	0.250000	0.436516	0.459503	5.265989
0.500000	0.500000	0.324660	0.333984	2.872115
0.500000	0.750000	0.436516	0.479427	9.830346
0.750000	0.250000	0.296484	0.292053	1.494481
0.750000	0.500000	0.227281	0.229458	0.957812
0.750000	0.750000	0.296484	0.302221	1.935071

Potential2d

File Help

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Potential Options

☒ Potential in a Rectangle

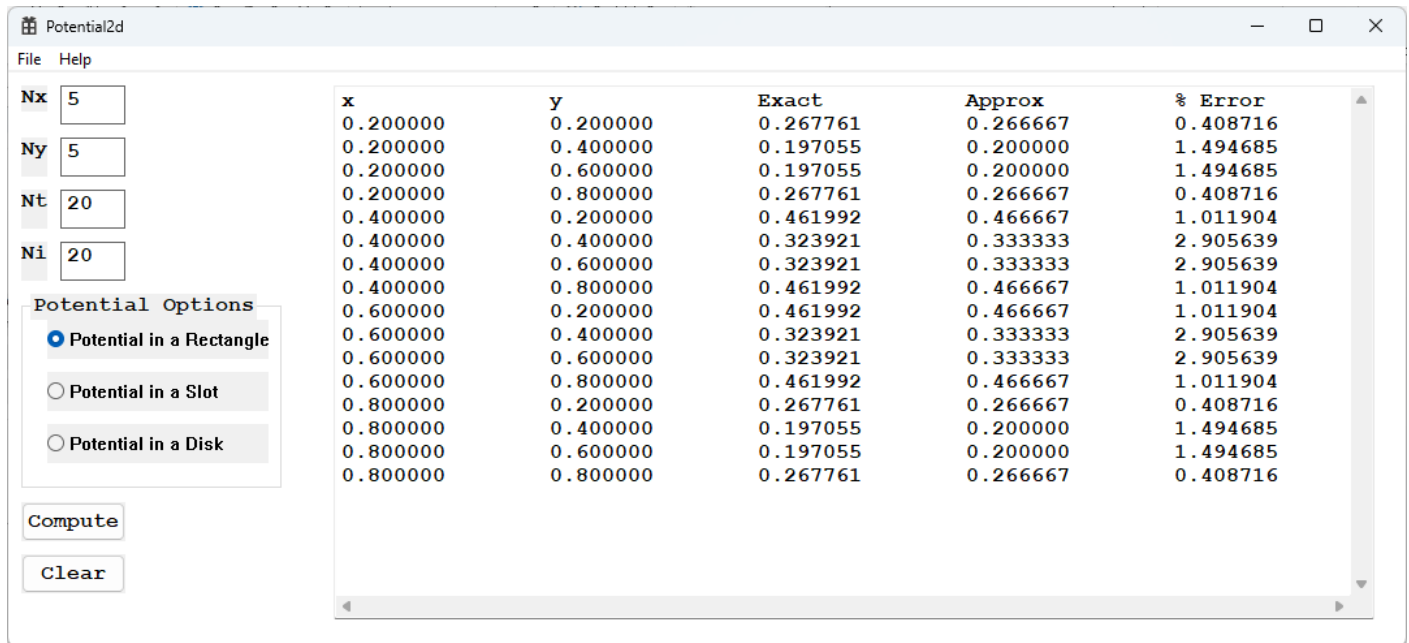
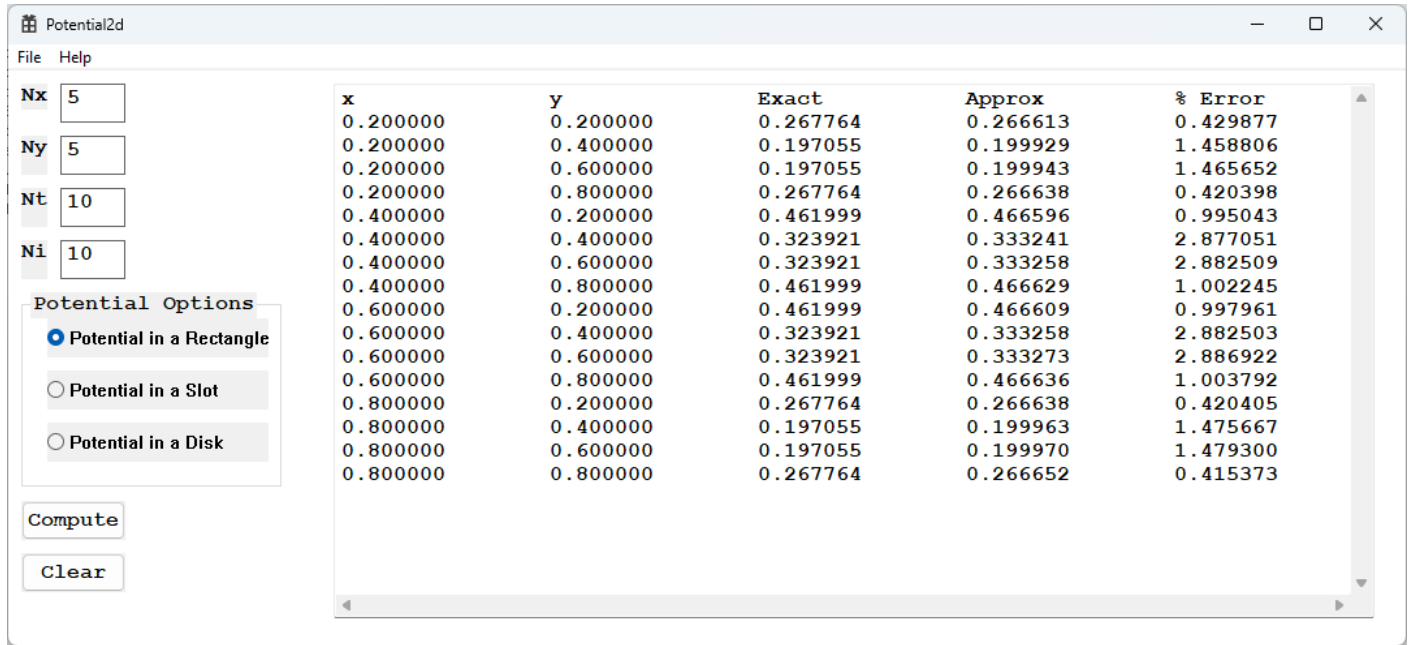
☐ Potential in a Slot

☐ Potential in a Disk

Compute

Clear

x	y	Exact	Approx	% Error
0.200000	0.200000	0.267591	0.268460	0.324900
0.200000	0.400000	0.197052	0.200808	1.905719
0.200000	0.600000	0.197052	0.199399	1.190814
0.200000	0.800000	0.267591	0.265866	0.644711
0.400000	0.200000	0.462152	0.463768	0.349657
0.400000	0.400000	0.323923	0.328908	1.538818
0.400000	0.600000	0.323923	0.329316	1.664827
0.400000	0.800000	0.462152	0.464488	0.505555
0.600000	0.200000	0.462152	0.461410	0.160485
0.600000	0.400000	0.323923	0.326985	0.945364
0.600000	0.600000	0.323923	0.328503	1.413804
0.600000	0.800000	0.462152	0.464355	0.476637
0.800000	0.200000	0.267591	0.263535	1.515632
0.800000	0.400000	0.197052	0.196365	0.349002
0.800000	0.600000	0.197052	0.197324	0.137689
0.800000	0.800000	0.267591	0.265420	0.811417



In my application I use tooltips for the four edit controls on the left-hand side of the interface. Nt is the number of terms used in the exact solution. Ni is the number of fixed-point iterations in the approximate solution.

The Laplace Equation for a slot is:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \forall x \in (0, a), \forall y \in (0, \infty)$$

$$u(x, 0) = f(x), x \in (0, a)$$

$$u(0, y) = g_1(y), \forall y \in (0, \infty)$$

$$u(a, y) = g_1(y), \forall y \in (0, \infty)$$

We insist on the solution to be bounded as y approaches infinity. We only solve one-half of solution which is given by the equations:

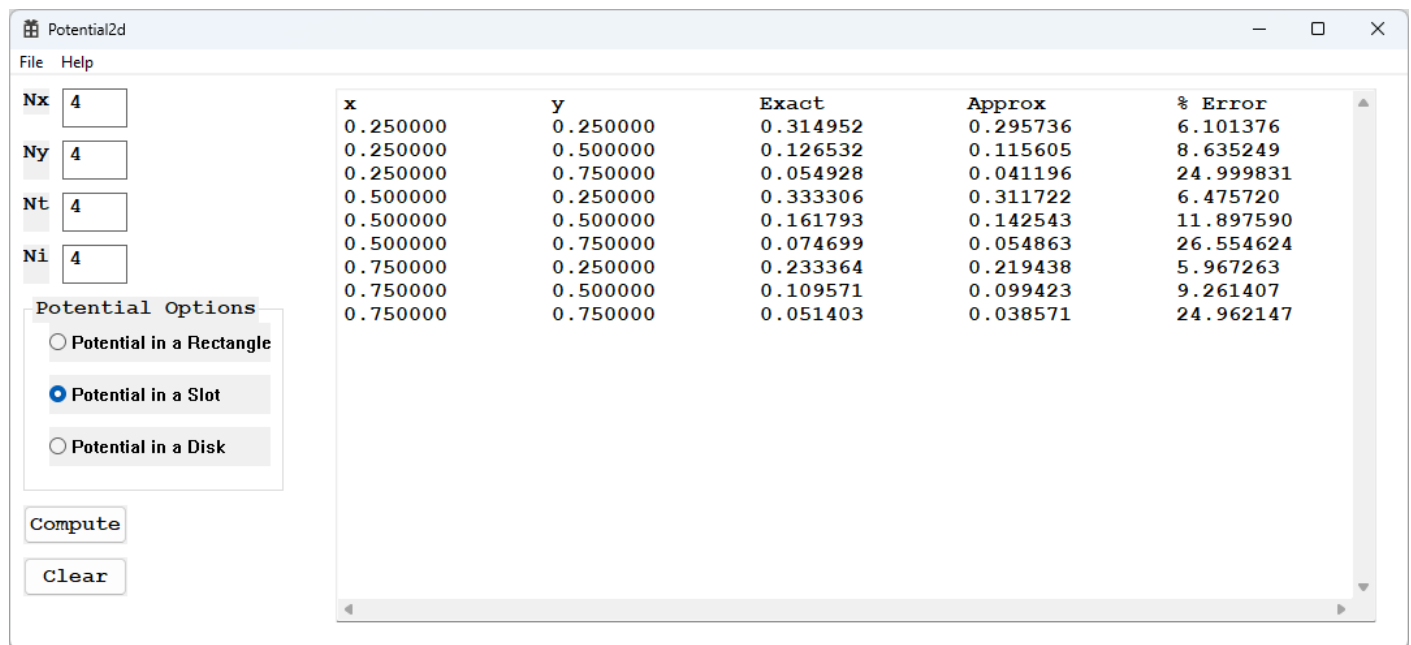
$$\nabla^2 u_1 = 0$$

$$u_1(x, 0) = f(x), \forall x \in (0, 1)$$

$$u_1(0, y) = 0, \forall y \in (0, \infty)$$

$$u_1(1, y) = 0, \forall y \in (0, \infty)$$

$$u_1(x, y) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \exp(-n\pi y)$$



Potential2d

File Help

Nx 5

Ny 5

Nt 5

Ni 5

Potential Options

☐ Potential in a Rectangle

☒ Potential in a Slot

☐ Potential in a Disk

Compute

Clear

x	y	Exact	Approx	% Error
0.200000	0.200000	0.347274	0.332818	4.162744
0.200000	0.400000	0.154073	0.145339	5.669006
0.200000	0.600000	0.075857	0.064326	15.201048
0.200000	0.800000	0.039019	0.025031	35.848455
0.400000	0.200000	0.400695	0.379037	5.405052
0.400000	0.400000	0.218857	0.196813	10.072599
0.400000	0.600000	0.116199	0.095716	17.627804
0.400000	0.800000	0.061577	0.039057	36.571672
0.600000	0.200000	0.350441	0.330275	5.754416
0.600000	0.400000	0.201418	0.180931	10.171717
0.600000	0.600000	0.110982	0.091585	17.477662
0.600000	0.800000	0.060072	0.038266	36.298991
0.800000	0.200000	0.231518	0.222579	3.861361
0.800000	0.400000	0.123066	0.114430	7.017824
0.800000	0.600000	0.067190	0.057109	15.002373
0.800000	0.800000	0.036565	0.023844	34.790323

Potential2d

File Help

Nx 5

Ny 5

Nt 10

Ni 10

Potential Options

☐ Potential in a Rectangle

☒ Potential in a Slot

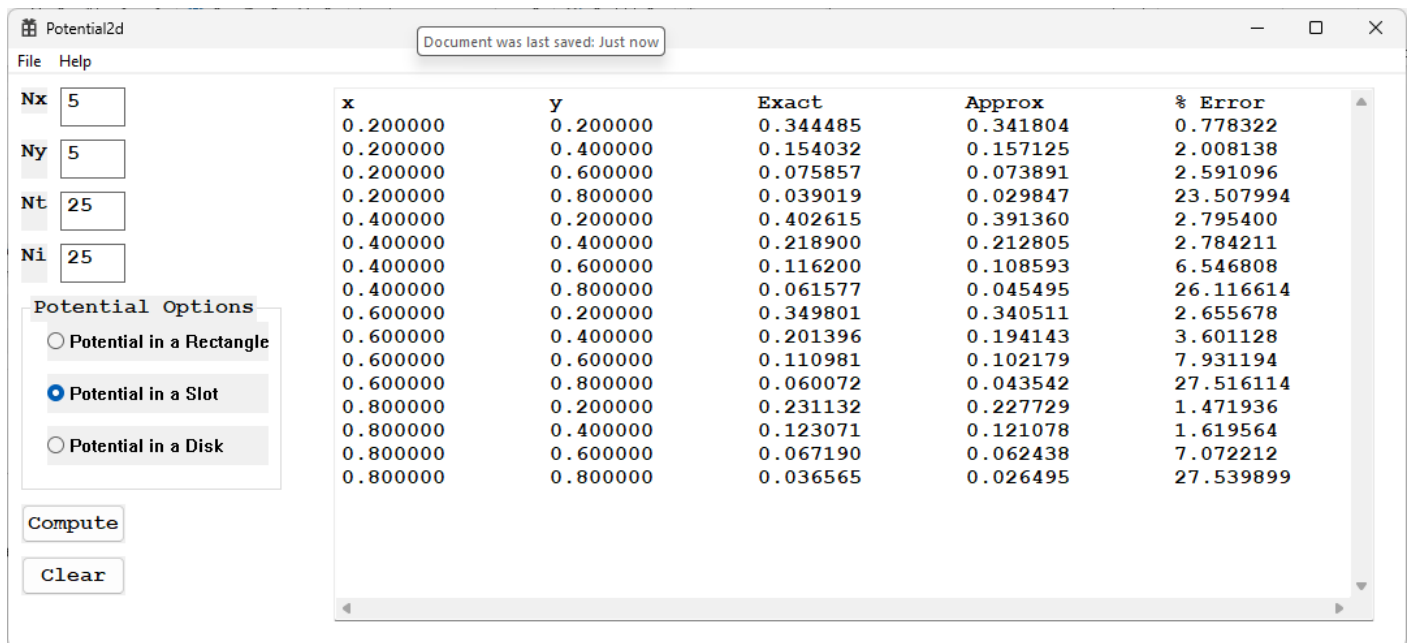
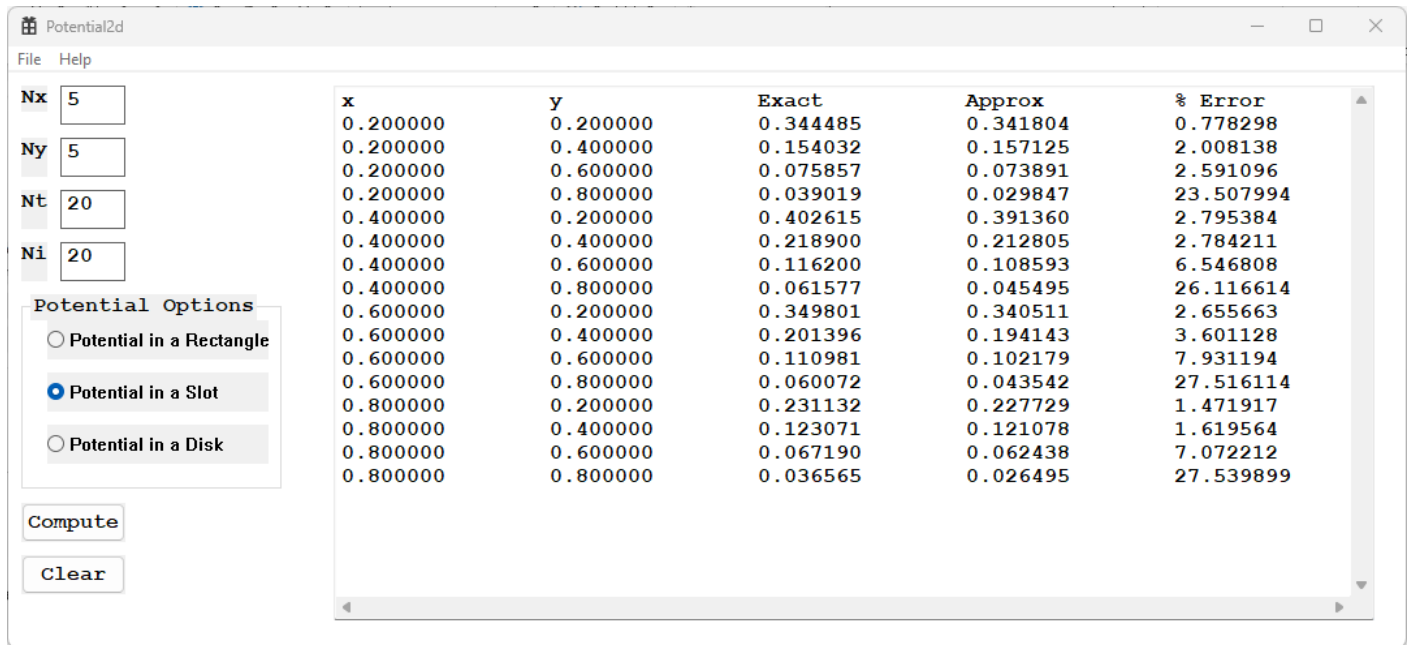
☐ Potential in a Disk

Compute

Clear

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x	y	Exact	Approx	% Error
0.200000	0.200000	0.344403	0.341668	0.794182
0.200000	0.400000	0.154032	0.156947	1.892816
0.200000	0.600000	0.075857	0.073747	2.780609
0.200000	0.800000	0.039019	0.029775	23.692208
0.400000	0.200000	0.402543	0.391182	2.822286
0.400000	0.400000	0.218900	0.212573	2.890440
0.400000	0.600000	0.116200	0.108405	6.708756
0.400000	0.800000	0.061577	0.045401	26.269418
0.600000	0.200000	0.349744	0.340367	2.680906
0.600000	0.400000	0.201396	0.193955	3.694538
0.600000	0.600000	0.110981	0.102027	8.068376
0.600000	0.800000	0.060072	0.043466	27.642834
0.800000	0.200000	0.231087	0.227658	1.484058
0.800000	0.400000	0.123071	0.120984	1.695986
0.800000	0.600000	0.067190	0.062362	7.185508
0.800000	0.800000	0.036565	0.026457	27.643992



Where we use the Fourier Sine Series. Finally, we compute the potential in a disk:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} = 0, \forall r \in (0, c), \theta \in (-\pi, \pi)$$

$$v(c, \theta) = f(\theta), \forall \theta \in (-\pi, \pi)$$

We use:

$$f(\theta) = \theta^2, \forall \theta \in (-\pi, \pi), c = 1$$

$$v(r, \theta) = a_0 + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} b_n r^n \sin(n\theta)$$

We use the standard Fourier Series coefficients.

Potential2d

File Help

Nx

Ny

Nt

Ni

Potential Options

☐ Potential in a Rectangle

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☒ Potential in a Disk

Compute

Clear

r	Theta	Exact
0.000000	0.000000	20.957094
0.000000	1.570796	20.360647
0.000000	3.141593	20.325375
0.250000	0.000000	21.204898
0.250000	1.570796	20.091325
0.250000	3.141593	19.967793
0.500000	0.000000	21.402050
0.500000	1.570796	19.850673
0.500000	3.141593	19.612228

Potential2d

File Help

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Potential Options

☐ Potential in a Rectangle

☐ Potential in a Slot

☒ Potential in a Disk

Compute

Clear

r	Theta	Exact
0.000000	0.000000	20.976933
0.000000	1.256637	20.597575
0.000000	2.513274	20.320610
0.000000	3.769911	20.479593
0.200000	0.000000	21.277426
0.200000	1.256637	20.524611
0.200000	2.513274	20.008208
0.200000	3.769911	20.264179
0.400000	0.000000	21.552310
0.400000	1.256637	20.446436
0.400000	2.513274	19.720618
0.400000	3.769911	20.021120
0.600000	0.000000	21.793555
0.600000	1.256637	20.367240
0.600000	2.513274	19.458071
0.600000	3.769911	19.764656

Potential2d

File Help

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Potential Options

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☒ Potential in a Disk

Compute

Clear

r	Theta	Exact
0.000000	0.000000	20.976936
0.000000	1.256637	20.597574
0.000000	2.513274	20.320607
0.000000	3.769911	20.479592
0.200000	0.000000	21.277576
0.200000	1.256637	20.524565
0.200000	2.513274	20.008055
0.200000	3.769911	20.264035
0.400000	0.000000	21.553707
0.400000	1.256637	20.445896
0.400000	2.513274	19.719094
0.400000	3.769911	20.019302
0.600000	0.000000	21.799310
0.600000	1.256637	20.364379
0.600000	2.513274	19.450880
0.600000	3.769911	19.754606

Potential2d

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Nx

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Potential Options

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Compute

Clear

r	Theta	Exact
0.000000	0.000000	20.976936
0.000000	1.256637	20.597574
0.000000	2.513274	20.320607
0.000000	3.769911	20.479592
0.200000	0.000000	21.277576
0.200000	1.256637	20.524565
0.200000	2.513274	20.008055
0.200000	3.769911	20.264034
0.400000	0.000000	21.553736
0.400000	1.256637	20.445884
0.400000	2.513274	19.719062
0.400000	3.769911	20.019261
0.600000	0.000000	21.799780
0.600000	1.256637	20.364099
0.600000	2.513274	19.450221
0.600000	3.769911	19.753600