

Three-Dimensional Heat Conduction Equation in Cartesian Coordinates (Rectangular Prism Which Includes a Cube)

Reference 1: For the one-dimensional heat conduction equation see - [Heat Conduction Equation -- from Wolfram MathWorld](#)

Reference 2: <https://www.wolframalpha.com/calculators/mathematics-calculus-integrals-definite-integral-calculator>

$$\frac{\partial U(x, y, z, t)}{\partial t} = \kappa \nabla^2 U(x, y, z, t)$$

$$\frac{1}{\kappa} \frac{\partial U(x, y, z, t)}{\partial t} = \nabla^2 U(x, y, z, t)$$

$$U(x, y, z, 0) = U_0$$

$$U(x, y, z, t) = X(x)Y(y)Z(z)T(t)$$

$$\frac{1}{\kappa} \frac{dT}{dt} = X''YZT + XY''ZT + XYZ''T$$

$$\frac{1}{\kappa T} \frac{dT}{dt} = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -\frac{1}{\alpha^2} - \frac{1}{\beta^2} - \frac{1}{\gamma^2}$$

$$\frac{1}{\kappa T} \frac{dT}{dt} = -\frac{1}{\alpha^2} - \frac{1}{\beta^2} - \frac{1}{\gamma^2}$$

$$\frac{dT}{T} = -\frac{\kappa dt}{\alpha^2}$$

$$\log T = -\frac{\kappa t}{\alpha^2}$$

$$T(t) = A_\alpha e^{-\kappa t / \alpha^2}$$

$$X''(x) + \frac{1}{\alpha^2} X(x) = 0$$

$$X(x) = B \cos\left(\frac{x}{\alpha}\right) + C \sin\left(\frac{x}{\alpha}\right)$$

$$X(0) = B = 0 \rightarrow B = 0$$

$$X(x) = A_\alpha C \sin\left(\frac{x}{\alpha}\right)$$

$$T(t) = A_\beta e^{-\kappa t / \beta^2}$$

$$Y(y) = D \cos\left(\frac{y}{\beta}\right) + E \sin\left(\frac{y}{\beta}\right)$$

$$Y(0) = D = 0 \rightarrow D = 0$$

$$Y(y) = A_\beta E \sin\left(\frac{y}{\beta}\right)$$

$$Z(z) = F \cos\left(\frac{z}{\gamma}\right) + G \sin\left(\frac{z}{\gamma}\right)$$

$$Z(0) = F = 0 \rightarrow F = 0$$

$$T(t) = A_\gamma e^{-\kappa t/\gamma^2}$$

$$Z(z) = A_\gamma G \sin\left(\frac{z}{\gamma}\right)$$

Suppose:

$$A_\alpha = A_\beta = A_\gamma = 1$$

$$X(a) = C \sin\left(\frac{a}{\alpha}\right) = 0 \rightarrow \frac{a}{\alpha} = n\pi \rightarrow \alpha = \frac{a}{n\pi}$$

$$Y(b) = E \sin\left(\frac{b}{\beta}\right) = 0 \rightarrow \frac{b}{\beta} = n\pi \rightarrow \beta = \frac{b}{n\pi}$$

$$Z(c) = G \sin\left(\frac{c}{\gamma}\right) = 0 \rightarrow \frac{c}{\gamma} = n\pi \rightarrow \gamma = \frac{c}{n\pi}$$

$$\begin{aligned} U(x, y, z, t) \\ = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_l E_m G_n \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi z}{c}\right) e^{-\kappa(l\pi/a)^2 t} e^{-\kappa(m\pi/b)^2 t} e^{-\kappa(n\pi/c)^2 t} \end{aligned}$$

$$U(x, y, z, 0) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_l E_m G_n \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi z}{c}\right)$$

$$U(0,0,0,0) = 0$$

$$U(a,b,c,0) = 0$$

$$\begin{aligned} & \int_0^a \int_0^b \int_0^c \sin\left(\frac{p\pi x}{a}\right) \sin\left(\frac{q\pi y}{b}\right) \sin\left(\frac{r\pi z}{c}\right) U(x, y, z, 0) dx dy dz \\ &= \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^a \int_0^b \int_0^c C_l E_m G_n \sin\left(\frac{p\pi x}{a}\right) \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{q\pi y}{b}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{r\pi z}{c}\right) \sin\left(\frac{n\pi z}{c}\right) dx dy dz \\ &= \left[\sum_{l=1}^{\infty} \frac{l}{2} \delta_{pl} C_l \right] \left[\sum_{m=1}^{\infty} \frac{m}{2} \delta_{qm} E_m \right] \left[\sum_{n=1}^{\infty} \frac{n}{2} \delta_{rn} G_n \right] = \frac{l}{2} C_p \frac{m}{2} E_q \frac{n}{2} G_r \end{aligned}$$

$$C_p = \frac{2}{a} \int_0^a \sin\left(\frac{p\pi x}{a}\right) U(x, y, z, 0) dx$$

$$E_q = \frac{2}{b} \int_0^b \sin\left(\frac{q\pi y}{b}\right) U(x, y, z, 0) dy$$

$$G_r = \frac{2}{c} \int_0^c \sin\left(\frac{r\pi z}{c}\right) U(x, y, z, 0) dz$$

Suppose the following:

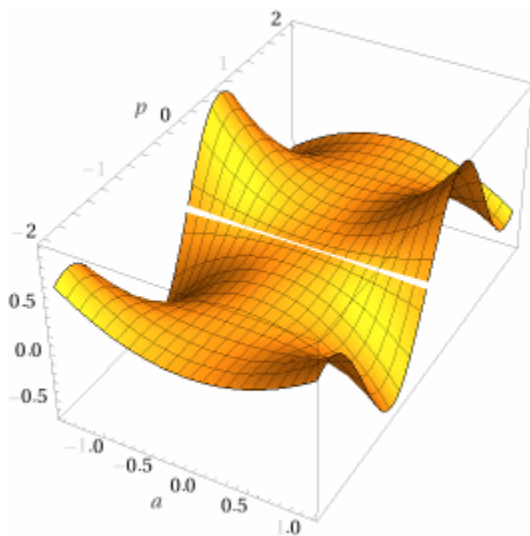
$$C_p = \frac{2}{a} \int_0^a x^2 \sin\left(\frac{p\pi x}{a}\right) dx$$

$$E_q = \frac{2}{b} \int_0^b y^2 \sin\left(\frac{q\pi y}{b}\right) dy$$

$$G_r = \frac{2}{c} \int_0^c z^2 \sin\left(\frac{r\pi z}{c}\right) dz$$

According to the Wolfram Definite Integral calculator:

$$\frac{2}{a} \int_0^a x^2 \sin\left(\frac{p\pi x}{a}\right) dx = \frac{2a^2((2 - \pi^2 p^2) \cos(\pi p) + 2\pi p \sin(\pi p) - 2)}{\pi^3 p^3}$$

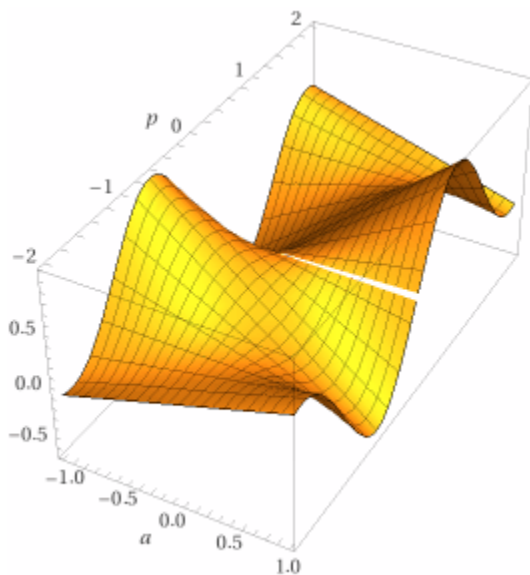


$$C_p = \frac{2}{a} \int_0^a x^2 \sin\left(\frac{p\pi x}{a}\right) dx = \frac{2a^2((2 - \pi^2 p^2) \cos(\pi p) + 2\pi p \sin(\pi p) - 2)}{\pi^3 p^3}$$

$$E_q = \frac{2}{b} \int_0^b y^2 \sin\left(\frac{q\pi y}{b}\right) dy = \frac{2b^2((2 - \pi^2 q^2) \cos(\pi q) + 2\pi q \sin(\pi q) - 2)}{\pi^3 q^3}$$

$$G_r = \frac{2}{c} \int_0^c z^2 \sin\left(\frac{r\pi z}{c}\right) dz = \frac{2c^2((2 - \pi^2 r^2) \cos(\pi r) + 2\pi r \sin(\pi r) - 2)}{\pi^3 r^3}$$

$$\frac{2}{a} \int_0^a x \sin\left(\frac{p\pi x}{a}\right) dx = \frac{2a(\sin(\pi p) - \pi p \cos(\pi p))}{\pi^2 p^2}$$



$$C_p = \frac{2}{a} \int_0^a x \sin\left(\frac{p\pi x}{a}\right) dx = \frac{2a(\sin(\pi p) - \pi p \cos(\pi p))}{\pi^2 p^2}$$

$$E_q = \frac{2}{b} \int_0^b x \sin\left(\frac{q\pi y}{b}\right) dx = \frac{2b(\sin(\pi q) - \pi q \cos(\pi q))}{\pi^2 q^2}$$

$$G_r = \frac{2}{c} \int_0^c x \sin\left(\frac{r\pi x}{c}\right) dx = \frac{2c(\sin(\pi r) - \pi r \cos(\pi r))}{\pi^2 r^2}$$

a = 1.5
b = 2.5
c = 3.5

k = 0.015

N = 5

C

0.7920949395798
-0.5179985843654
0.3476467357979
-0.2661802978964
0.2126015657637

E

3.9012284517178
-2.3688763130395
1.6181468231367
-1.2286580338090
0.9861394861651

G

10.8174642236168
-6.4861349339675
4.4443613031354
-3.3696799361453
2.7068666968539

U

6.5845606817686
-6.3190422999084
4.2520838282102
-1.9425469310477
0.0000000000000

C:\Users\James\OneDrive\HeatConductionCube\x64\Debug\HeatConduct
ionCube.exe (process 17172) exited with code 0 (0x0).

Press any key to close this window . . .